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AND ITS PRACTICAL APPLICATION TO THE
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It would be hopeless to attempt to mention the erroneous statements made in books upon Photography.

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GEOMETRICAL OPTICS

CHAPTER I

INTRODUCTORY

LIGHT is the agency by means of which the eye enables us to perceive and appreciate the existence and some of the qualities of bodies at a distance from the eye.

The physical theories of light have been various, but a work upon geometrical optics has only to deal with those geometrical matters connected with light which are true under any of these theories.

Any substance through which light can be exercised is called a transparent medium. Air, glass, and water are well-known cases of such media. Substances which do not transmit light through them are called opaque. Some bodies shed light independently of others. This property can be tested in many cases by bringing the bodies into a dark room, when they will remain visible.

Red-hot bodies, lamps, flames, glowworms, and the tips of phosphorous matches, also certain minute bodies in disturbed sea-water, are of this class. So too are a very large number of organic substances, which at an extremely low temperature have been exposed to light for a short time.

Such bodies are called sources of light. In opposition to this class of bodies is that of those which are visible only when light falls upon them from elsewhere. Such bodies in a dark room are invisible, and most bodies at ordinary temperatures have this property. An unpolished ivory counter may be kept in mind as a type of such bodies. But by whichever of these processes a body may be visible, the light that is shed by it obeys the same geometrical laws.

The first and most important of these laws is that light is shed *in all directions* from a visible body, and *in straight lines*, so long as the medium through which it has to pass remains uniform.

Thus an ivory counter or a glowworm in the middle of a table is visible to all eyes which are above the horizon of the table.

Lamps and other sources of light cast shadows of opaque bodies of shapes which could only be produced by this law of light propagation.

Thus a small visible body situated in a uniform medium, as air, may be considered as emitting rays which proceed in straight lines in all directions to the boundary of the medium wherever that may be situated. The boundary is then said to be illuminated, and light may proceed from its various points as before from the original object.

The boundary of the medium may be an eye turned towards the small body, in which case the light enters the eye and affects its nerves in a way which experience will have instructed it to interpret as due to the presence of the small body in the direction of the straight line drawn from the eye to that body, *i. e.* in a direction exactly opposite to that in which the light was travelling when it encountered the eye.

The aperture of the eye is small, the pupil being the unobstructing portion of it through which alone the light is allowed to proceed to the nerves beyond. So the total quantity of light received by the eye from any one small object is comprised in the small-angled cone whose vertex is the small object, and whose base is the pupil of the eye.

Such a small cone of light-filled space is often called a pencil of light, and in the case supposed when the light is being propagated *from* the vertex, it is called a *divergent* pencil, for any two of the straight rays constituting the pencil are becoming further and further removed one from the other as the propagation goes on.

It will be seen that although all light coming directly from ordinary natural objects, and keeping in one uniform medium, consists of such divergent pencils, still there are means of changing these divergent rays (1) into parallel rays, in which case the pencil is called a parallel pencil, (2) into rays which, as the propagation goes on, tend *towards* a point, forming a *convergent* pencil. If a convergent pencil is allowed to reach the point to which the rays converge, they will pass through it; and beyond it will form a *divergent* pencil again: and if these are then received by an eye, the impression produced will be that of the original object situated at the point of this last divergent pencil. Such a point is called a real image of the original point. It is of course quite independent of the existence of any eye to appreciate it.

It may also happen that a pencil falling upon the boundary of the medium may in its subsequent path *diverge* at once as from another point. The rays have in this case not really passed through that point, but will pursue a path in future as if they had done so. Such a point is called a *virtual* image. The images ordinarily formed by flat wall-mirrors are all of this character.

And again, rays which are convergent, and therefore if allowed to proceed far enough would form a real image, may be intercepted before that point is reached and diverted elsewhere. The point is then again called a *virtual* image. The light does not really reach it. Thus the test of *virtuality* or *reality* in an image is the following :—

Do the rays ever pass through any of the vertices of the various cones which form the portions of the path of light?

If they do, that point is a real image.

If they do not, that point is a virtual image.

The eye being a natural organ developed for the purpose of using *without further assistance* the divergent rays which occur under ordinary circumstances, is adapted for the reception of divergent pencils and of parallel pencils which may be regarded as divergent from very distant objects.

Convergent rays in general are useless for direct reception upon the eye.

From what has been said it will be seen that if rays start from an object, any selected pencil of the light may undergo such changes as will involve the formation of images at a number of points, and if the pencil at last enters an eye in a divergent or parallel condition, the last image, whether real or virtual, will be that which produces the sensible effect of vision. The position of this image and the direction of the pencil which leaves it, are therefore matters of the highest importance in dealing with optical instruments.

The direction of the pencil may be taken to be that of its central ray, and it may therefore be determined by following out the path of that ray.

There are only two ways by means of which a ray of light may have its course definitely turned from that which it has been pursuing. The first of these ways is called reflection, and it occurs when the boundary of the medium is a polished surface, in which case some portion of the incident light is always bent back so as again to traverse the original medium, *i. e.* it will have its path on the same side of the tangent plane to the boundary at the point of incidence. The bent back portion of the ray is called the reflected ray, and its path is governed by the simplest geometrical laws, for the expression of which the following definitions may be advantageously employed.

The angle which the reflected ray makes with the normal at the point where the bending takes place is called the angle of reflection.

The angle which the incident ray makes with the normal at the point where the bending takes place is called the angle of incidence.

The laws of reflection may then be stated as follows: The angle of reflection is equal to the angle of incidence, and the incident and reflected rays and the normal are in one plane.

A polished surface employed to reflect light is called a mirror. A surface of any shape may receive a polish, but practically mirrors are made of plane or spherical surfaces, and very exceptionally of paraboloids and ellipsoids, and cylinders.

The second method by which a ray is bent is called refraction, and it occurs when the ray of light encounters the boundary, crosses to the other side, and then traverses the medium on that side.

What happens in this case will depend upon the nature of the media, and the phenomena may be complex; but if the media are isotropic (having all directed properties independent of direction), such as water and other liquids, air and other gases, and simple solids, such as uniform unstained glass, the phenomena are greatly simplified and what is called simple refraction takes place. The following definition is required:—

The portion of the ray within the second medium called the refracted ray, makes an angle with the normal which is defined as the angle of refraction.

The laws of simple refraction are, that the sines of angles of incidence and refraction for the same two media are always in a constant ratio, and that the incident and refracted portions of the ray and the normal at the point of incidence are in one plane.

CHAPTER II

REFLECTION OF LIGHT—MIRRORS

LET A be a small object shedding light, and let M N be a plane mirror, the polished surface being turned towards A.

Suppose the plane of the paper to contain A and the perpendicular A B, which can be dropped from A upon the mirror, as well as any ray A P coming from A and encountering the mirror at some point P.

Draw P D the normal at P, and therefore parallel to B A, and make the angle D P E equal to the angle D P A, but on the opposite side of the normal.

Then P E must also lie in the plane of the paper, that is in the plane containing A P, P D; and as further D P E is equal to D P A, therefore by the two laws of reflection, P E must be the course of the ray under consideration after reflection.

Produce E P, A B until they meet in C. Then the two triangles A B P, C B P have the two angles at B equal, and the two angles at P equal, and further the side P B is common to the two, therefore the two triangles are equal in the remaining sides and angles, or B C is equal to B A. The ray will travel after reflection away from C, whose position depends only on that of A relative to the mirror, and not upon the particular ray considered, and hence *all* rays which have suffered reflection at the mirror will subsequently proceed in a direction away from C. Thus every reflected pencil must after reflection proceed as from C, or this point is, according to definition, a virtual image of A. Any eye which receives any reflected pencil perceives this virtual image at C, which is a spot fixed by the condition that it must be as far behind the mirror as A is in front of it, and both points are in the same perpendicular A B.

For every point A there is an image at point C, following this simple rule. But any finite object may be considered to be made up of points, each of which will have its separate point image, forming together what may be called the image of the original object. Thus P B C may be called the image of the Δ P B A, and P C the image of the line P A.

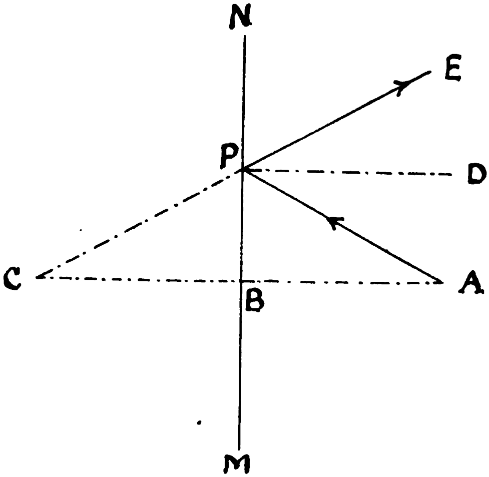

Fig. 1.

Such an image obeys necessarily the following laws. If A move parallel to the mirror, the path of C will be parallel to that of A; but if A move along a normal, C will move in an opposite direction in space. If the motion of A is compounded of these two sorts, that of C will be compounded of two, following the above rules. Thus suppose M N is in the direction of south to north, and A B in that of east to west,

then if A move northwards C will move northwards
if A „ eastwards C „ „ westwards
if A „ northeastwards C „ „ northwestwards.

A plane mirror may be employed when it is desired to give a ray any proposed deviation from its previous path at any point of its course. Suppose AP a ray and it be desired at the point P to change the course of the ray to PE.

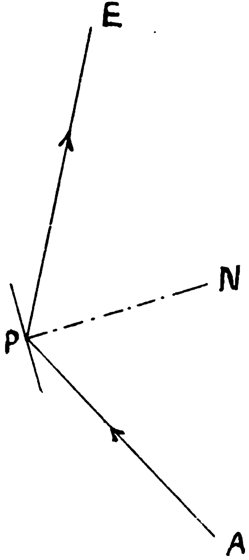

FIG. 2.

Bisect the angle APE by the line PN, and place at P a mirror whose normal coincides with PN as in the figure. If the mirror at P be made to turn through an angle round an axis through P at right angles to the plane of reflection, PN will move relatively to PA through this anglé, but EPN must remain equal to APN: therefore PE must move relatively to PA, through twice this angle. That is, the plane containing the reflected ray and the axis of rotation moves twice as fast in angle as the mirror moves. But if the axis of rotation of the mirror coincides with the incident ray, the plane containing the reflected ray and the axis of rotation moves at the same angular rate as the mirror.

If two plane mirrors are inclined one to another, a ray of light may be reflected backwards and forwards between the mirrors several times, images being formed by each reflection. The number of these reflections is limited by the following rule.

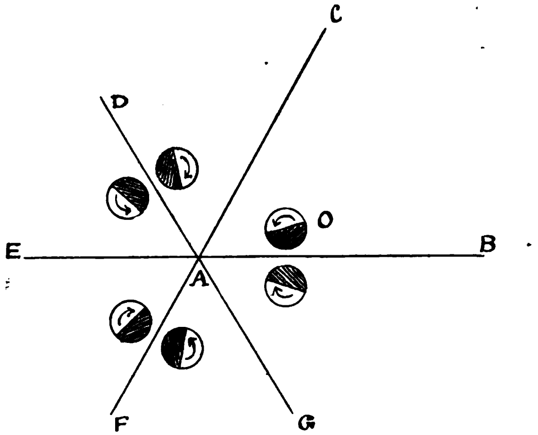

FIG. 3.

If θ be the angle between the mirrors in circular measure, then $\frac{\pi}{\theta}$ is the total number of possible reflections of any one ray. But as rays fall, first some on one mirror and some on the other, there will be $\frac{\pi}{\theta}$ images formed for each set of rays, thus forming $\frac{2\pi}{\theta}$ images. When, however, $\frac{\pi}{\theta}$ is a whole number, the last images of each set coincide, and to the eye only $\frac{2\pi}{\theta} - 1$ images will appear.

Now $\frac{2\pi}{\theta}$ is the number of angular compartments, each of the value θ, into which 2π can be divided. Hence any object situated between the mirrors forms a symmetrical group of $\frac{2\pi}{\theta}$ members, each member occupying one compartment. Of these $\frac{2\pi}{\theta} - 1$ are images, and of them $\frac{\pi}{\theta}$ will be left-handed, $\left(\frac{\pi}{\theta} - 1\right)$ right-handed. This may be represented by Fig. 3, in which the object is represented as a counter having a black and a white limb turned respectively to the two mirrors B, A C, reflecting of course on the sides turned towards each other.

The compartments are here six in number as the angle B A C is $\frac{\pi}{3}$.

C A B contains the object O.

C A D and B A G contain images formed by one reflection, and therefore left-handed.

P A E and G A F contain images formed by two reflections, and therefore right-handed.

E A F contains an image formed by three (the greatest number possible) reflections, and therefore left-handed.

If a motion of rotation be given to the object, the left-handed images will be observed to rotate in a reverse direction, but the right-handed images will have a direct rotation.

The image in the compartment B A G could not be seen by any eye not in front of the mirror A B. It is caused by one reflection only, and therefore that must be by the mirror A B.

Similarly the image in D A C must be caused by reflection once in the mirror A C.

Similarly the eye that sees the image in D A E must be situated in the region C A G, and that which sees an image in F A G must be situated in B A D. And finally the compartment E A F is only visible to an eye in B A C.

If $\frac{\pi}{\theta}$ is a whole number larger than four, the extension of the foregoing rule to meet the case is clear. For each reflection the region of visibility is restricted by one compartment, but always contains the object compartment.

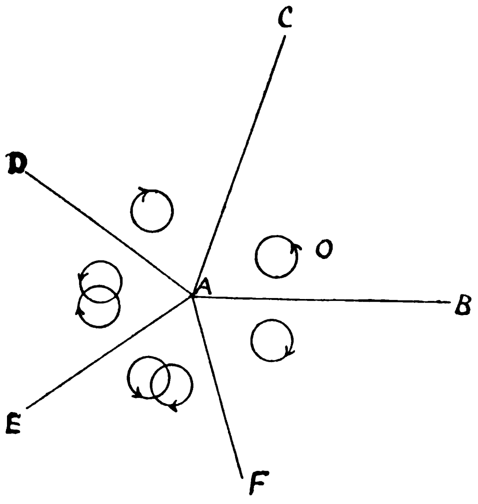

FIG. 4.

When $\frac{2\pi}{\theta}$ is an odd number the facts are somewhat different. The whole number of compartments being odd, the image compartments are even, and there will be a division between compartments exactly opposite the centre of the object compartment.

The figure illustrates the case of $\frac{2\pi}{\theta} = 5$.

The object is O, purposely placed a little out of the centre line of the compartment B A C contained by the mirrors.

In CAD, BAF there are left-handed images formed by one reflection only, in AC and AB respectively.

In each of the remaining two compartments DAE, EAF, are shown two images, one right-handed and one left-handed.

The two right-handed images are the most easily and directly seen, thus presenting the unusual appearance of two images of the *same* sort in adjoining compartments.

The right-handed image in DAE is formed by two reflections, of which the first is in AB, the second in AC.

The right-handed image in EAF is formed by two reflections, of which the first is in AC, the second in AB.

The left-handed image in DAE is the result of three reflections, of which the first and last are in AB, the second in AC.

The left-handed image in EAF is the result of three reflections, of which the first and last are in AC, the second in AB.

These images are only perfect if the object is entirely on one side of the plane bisecting BAC.

In the figure the *larger half* of the object is situated towards the mirror AB. In this case it is the *larger half* only of the left-handed image in DAE which can be seen, and that only by an eye in that half of CAB nearer to AB.

And the *smaller half* only of the left-handed image in EAF is visible, and only by an eye in that half of CAB which is nearer AC.

It follows that of these images no two corresponding points can be seen at the same time.

If the value of $\frac{2\pi}{\theta}$ is as above 5, or 9, 13, etc., the above remarks are true, but for the values 3, 7, 11, etc., the imperfect images will be the result of an even number of reflections, and will therefore be right-handed.

CHAPTER III

THE SIMPLE REFRACTION OF LIGHT AT PLANE SURFACES

If A B is a ray of light emanating from a small object at A, and encountering the boundary of the medium F B H, here supposed to be a plane, at B, penetrates through this boundary to a region occupied by a second medium, it will pursue a course B C according to a law which has already been pointed out, viz. that the sines of the two angles which the ray makes with the normal D B E are always in constant ratio. Expressed formally this is equivalent to the equation—

$$\frac{\text{Sine DBA}}{\text{Sine CBE}} \text{ is constant } = \mu.$$

μ is generally taken as the ratio of the larger sine to the smaller, so that it appears as a number greater than unity.

One plane contains the lines A B, B C, D E, and A B, B C are on opposite sides of the normal D B E.

Now μ, which is called the index of refraction for the two media, though constant for different angles of incidence, is not constant for rays of different colours.

For the same medium the index decreases as we deal with rays ranging from violet to red.

For general purposes it is quoted for a particular sort of yellow light which is produced when sodium is present in the source of light. This is found to be almost universally the case in all ordinary sources.

For particular purposes the index of the rays employed must be taken.

It follows that an eye at or near C will receive rays of different colours in slightly different directions, and if this is to be avoided (for the sake of clear vision), light of one sort only at a time must be used. Such light is called monochromatic light.

Supposing then that monochromatic light of index μ is employed, there still exist reasons why no definite image of A will be seen by an eye situated in the line BC.

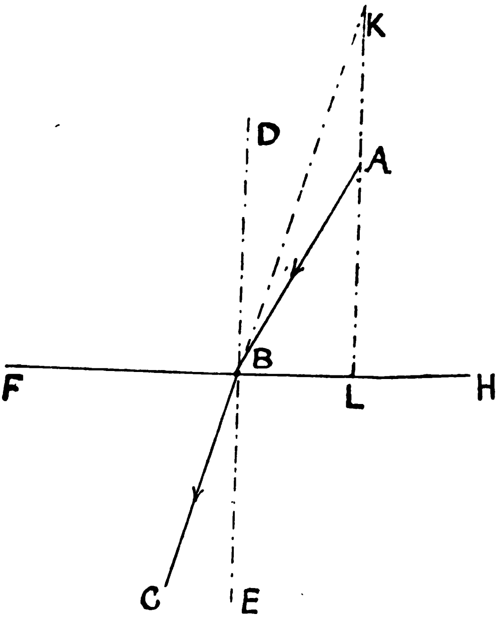

FIG. 5.

From A draw a perpendicular AL to the surface and produce CB until it cuts LA, produced or not, as the case requires, in K.

Vision being the result of collecting together a number of rays emanating from the same point and nearly coinciding with a centre one, we must suppose rays slightly displaced from A B in every possible direction to be present. Let the displaced ray be such as would result from revolving A B through a very small angle round A L as an axis. Then clearly the refracted ray produced backwards would again pass through K, and this point would therefore be the image of A for rays undergoing such displacement.

But now suppose the displacement made in an angular direction at right angles to the former one, *i.e.* round an axis through A at right angles to the plane containing D E F H. Then the ray considered will still be in that plane, but make a slightly different angle of incidence with the boundary plane. The refracted ray produced backwards will meet B K in some point removed from K by a distance which depends upon the angle of incidence. When the incidence angle is zero, the points indicated above coincide, and a genuine image is produced at the position which K will under those circumstances occupy.

This position is obtained from the consideration that—

$$\mu = \frac{\text{Sine DBA}}{\text{Sine EBC}} = \frac{\text{Sine LAB}}{\text{Sine LKB}} = \frac{\text{BL}}{\text{BA}} \times \frac{\text{BK}}{\text{BL}} = \frac{\text{BK}}{\text{BA}}$$

Now when the incidence angle is zero, B will coincide with L, and therefore B K becomes L K, and B A becomes L A.

$$\therefore \ \mu \text{ is equal to } \frac{\text{LK}}{\text{LA}}$$

$$\text{and LK} = \mu \text{ LA}.$$

It will thus be understood why in any apparatus where refraction is employed, and when a definite image for the sake of clear vision is to be produced, only small angles of incidence can be permitted. The only exception is when light is passed through a plate which is defined to be a mass of refracting material contained between two parallel planes.

It is shown in discussing the physical causes of refraction that the index of refraction is the relation of the velocities of light propagation in the two media. If therefore θ_1 be the angle of incidence and θ_2 the angle of refraction, and if v_1, v_2 are the velocities at which light is propagated in the two media respectively, then $\sin\theta_1 = \frac{v_1}{v_2}\sin\theta_2$

$$\text{or } \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

If, therefore, the second medium be followed by a third medium from which it is separated by a plane boundary parallel to the first boundary, θ_2 is the angle of incidence at this second boundary, and θ_3, the angle of refraction must be such that—

$$\frac{\sin\theta_2}{v_2} = \frac{\sin\theta_3}{v_3}$$

where v_3 is the velocity of light propagation in the third medium.

But if the third medium had been substituted for the second we should have had—

$$\sin\theta_1 = \frac{v_1}{v_3}\sin\theta_3$$

$$\text{or } \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_3}{v_3}$$

Thus the angle which the light makes in the third medium with the normal to its boundaries is independent of the existence of the second medium. This proposition may obviously be generalized.

If media are disposed in parallel plates the direction of the light in any one plate is independent of the rest.

Supposing we were justified in assuming that the atmosphere is disposed in horizontal films each of uniform character, then a telescope placed in an air-tight case whose top is formed of a horizontal plate of glass, and from which all air has been withdrawn, would have to be pointed in the true direction of a star in order to receive its light.

$\frac{\text{Sin }\theta_1}{v_1}$ is an intrinsic function of what is taking place in the particular medium indicated by the subscript. v_1 may be called the absolute index of refraction for the medium in question, as it is independent of the nature of the medium beyond the boundary.

Sometimes, especially for the purposes of registration of indices, the medium from which the light comes in crossing the boundary is taken to be the luminiferous ether, unencumbered by the presence of grosser matter. In such a case the index of any substance, say glass, is understood to be the ratio of the sine of the angle of incidence, supposed to take place in ether, to the sine of the angle of refraction of the same ray when it passes into glass.

The following table gives a few values of μ for yellow light :—

SOLIDS.		Turpentine	1·46
Diamond	2·42	Sulphuric acid	1·44
Flint-glass	1·6	Nitric acid	1·42
Crown-glass	1·5	Ether	1·35
Ice	1·31	Water	1·33
LIQUIDS.		GASES.	
Bisulphide of carbon	1·63	Air	1·0003
Glycerine	1·47	Hydrogen	1·00014

It will be noticed that the index for gases differs very slightly from unity; it therefore follows that the above numbers may be taken as true for the passage of rays from air to any of the liquids and solids mentioned. The error will not amount to one of 3 in 10,000.

Sometimes the statement is made that refraction towards the normal takes place when light passes from a rarer into a denser medium. Though this is generally true, it is not so universally, for turpentine, which floats upon water, and therefore is not so dense as water, has an index not only higher than water, but higher than the heavier nitric and sulphuric acids. In such statements as those referred to, the word *dense* must be taken as meaning *optically dense*, *i.e.* as allowing a comparatively slow velocity of light propagation.

The laws of simple refraction may be illustrated very conveniently by a mechanical model.

Let four equal rods be jointed together at the ends so as to form a rhombus. Place them in the form of a square whose plane is vertical, two opposite sides being horizontal, and pin them at the middle points A and B of these sides to a vertical blackboard. Let C D be the middle points of the vertical links of the framework. Bisect A B in O and draw a horizontal line on the blackboard through O.

Let E be the right-hand top corner, and let F be the left-hand bottom corner of the square. Take P in C E such that C E : C P :: $\mu : 1$ and take B Q equal to C P. Put pegs into the link-work at P and Q, and into the blackboard at O.

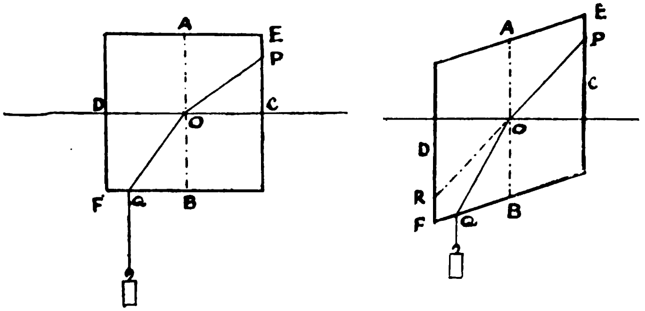

FIG. 6.

Fasten a string to the peg P, let it pass over O and Q and be kept tight by a weight suspended at the end.

Now let the link-work move round the pins at A and B. Then in any position in which P is above the horizontal line through O, P O will represent a ray incident upon the horizontal surface through O, and O Q will represent the same ray refracted in the medium below O, the index of refraction being μ or the relation of C E to C P. For in the triangle B O Q (in the second woodcut)

$$\frac{\text{Sine } OQB}{\text{Sine } BOQ} = \frac{OB}{BQ} = \frac{CE}{CP} = \mu$$

and from considerations of symmetry the angle O Q B is equal to the angle O P C, which is equal to P O A.

$$\therefore \text{Sine } POA = \mu \text{ sine } BOQ.$$

Produce P O to meet the link D in R. Then clearly the angle R O Q is the deviation the ray has undergone owing to the refraction.

Suppose the rhombus folded into a vertical straight line, which will be A B produced. This represents vertical incidence, and as every point in the whole link-work is in the straight line A B there will be no deviation. In this position R and Q will coincide and the distance of Q from O will be equal to (O B + B Q) since O B Q will form a straight line.

Now in opening out the rhombus from this linear condition, R Q will continually separate and O Q will continually become smaller. Therefore the relation $\frac{RQ}{2} : OQ$ on both accounts becomes greater. But this relation is the sine of half the deviation. Hence the deviation increases as the angle of incidence increases.

If E F are joined by a straight line, it is evident that P O and Q O make equal angles with this line, and each of those angles is half the deviation, and the line E F makes with the normal an angle which is the mean arithmetically of the two angles which the incident and refracted portions of the ray make with it.

If γ is the value of this angle and Δ is the deviation, then $\gamma + \frac{\Delta}{2}$ is the angle of incidence, and $\gamma - \frac{\Delta}{2}$ is that of refraction. Hence $\sin\left(\gamma + \frac{\Delta}{2}\right) = \mu \sin\left(\gamma - \frac{\Delta}{2}\right)$.

Expanding $\sin\gamma \cos\frac{\Delta}{2} + \cos\gamma \sin\frac{\Delta}{2}$

$$= \mu\left(\sin\gamma \cos\frac{\Delta}{2} - \cos\gamma \sin\frac{\Delta}{2}\right)$$

$$\overline{\mu - 1} \sin\gamma \cos\frac{\Delta}{2} = \overline{\mu + 1} \cos\gamma \sin\frac{\Delta}{2}.$$

$\tan \gamma = \left(\frac{\mu+1}{\mu-1}\right) \tan \frac{\Delta}{2} = \left(\frac{v_1+v_2}{v_1-v_2}\right) \tan \frac{\Delta}{2}$ where v_1 and v_2 are the absolute indices of refraction.

As this equation is true for finite values it will also be true in the limit, where we are dealing with a continuously varying medium. In such a case γ is the angle of incidence and Δ is its small increase through an element or $d\gamma$, and $v_1 + v_2$ may be written 2μ; and $v_1 - v_2$ may be written $-dv$ or $-d\mu$.

Hence, substituting θ for γ, the equation may be transformed into the form,

$$\tan \theta = -\frac{\mu}{d\mu} d\theta \text{ or } \frac{d\theta}{d\mu} = -\frac{\tan \theta}{\mu}$$

which is the general differential equation of a refracted ray, passing in a space in which the index of refraction varies continuously.

Critical angle and Total Internal reflection.

If a ray in a less refracting medium makes an angle of incidence equal to a right angle, the sine of the angle of refraction must be equal to $\frac{1}{\mu}$. This angle is called the critical angle, and for the following reason. If the ray is passed backwards along its path it will make in the more refracting medium an angle of incidence equal to the critical angle, and will, after emergence in the less refracting, pursue a path parallel to the bounding surface; but this angle of incidence is less than 90°. It can therefore be increased, in which case there is no solution for the angle of refraction. The ray will not emerge, but will remain within the more refracting medium *following the laws of reflection.* This phenomenon is called total internal reflection, and can only occur in a medium separated by the boundary where the incidence takes place from a *less* refracting medium.

As an example, the value of μ for air and glass being $\frac{3}{2}$, the critical angle has for its sine the value $\frac{2}{3}$, and the angle itself will therefore be 41° 48′ 40″.

If then a ray within the glass makes an angle of incidence greater than this angle, the ray will be reflected at the surface, remaining within the glass. It is on this account that an isosceles right-angled prism of glass, receiving a beam of light at vertical incidence on any one of its sides, may be used as a reflector. If the side which first receives the light is one of the shorter sides, total internal reflection will take place on the longer side, but if this latter be the first to receive the light, total internal reflection takes place at each of the short sides.

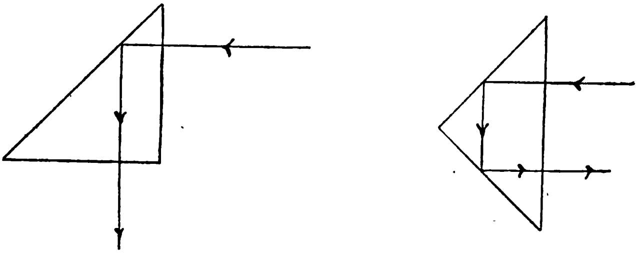

FIG. 7.

If in the former of these two cases the light be finally received on a screen, a white patch will of course result; but if a drop of water is allowed to dribble down the hypotheneuse of the prism, an absorption spot will at once appear in the patch of light, thus showing experimentally that it was a physical property of the medium (air) outside the boundary that prevented the light from passing through it.

CHAPTER IV

REFLECTION AT SPHERICAL SURFACES

LET O be a small object emitting light: and let C be the centre of a sphere A P upon which the light falls, whose surface is in a condition (polished) to reflect light which encounters it. Either the convex or concave surface may be polished: in the figure the convex surface forms the mirror; and O is placed further from C than the radius C A of the sphere. Let C O be joined, cutting the sphere in the point A, and let O P be a ray of light meeting the surface of the sphere in P. Join C P and produce it to any point K, and draw P E making an angle K P E equal to K P O, but on the opposite side of P K.

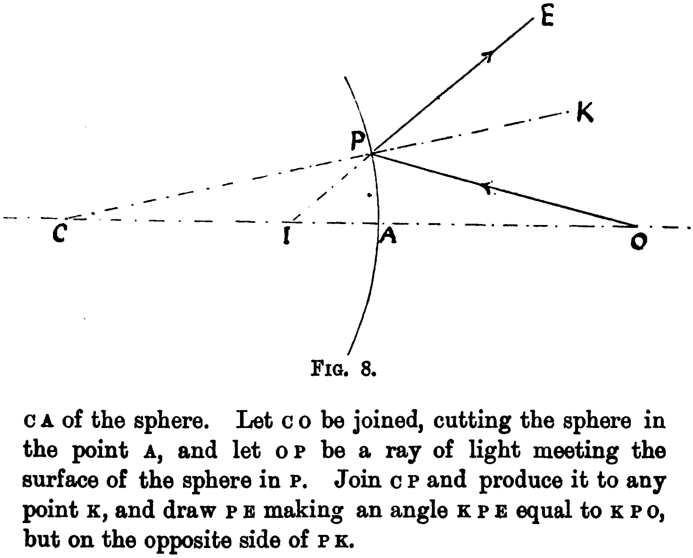

FIG. 8.

Then by the laws of reflection P E is the reflected ray.

Produce E P backwards to cut C A in I.

This point will not be fixed for all rays emanating from O, as was the case with plane mirrors. Hence images in spherical mirrors do not appear stationary when the eye is moved. If OP is considered the central ray for a small pencil, the image for such a pencil would be the point where the production backwards of the different rays meet in a point.

Now, if a ray is displaced from the position OP by a revolution round CO, such a ray will necessarily pass through I.

But if the displacement is effected by revolution round an axis through O at right angles to the plane CPO, the reflected ray when produced backwards will not cut PI in I, unless P is indefinitely close to A, *i.e.* the pencil must make vertical incidence upon the mirror. Under such circumstances P must be taken close to A, and then I will be the image of O. This is expressed by saying that the ultimate position of I when P moves up to A is the image of O. We proceed to determine its position.

In the Δs CPI, CPO

$$\frac{CI}{IP} = \frac{\text{sine } CPI}{\text{sine } PCA}; \quad \frac{CO}{PO} = \frac{\text{sine } CPO}{\text{sine } PCA}$$

but sine CPI is equal to sine CPO as the sum of the two angles is equal to 180°.

$$\therefore \frac{CI}{IP} = \frac{CO}{PO} \text{ and in the limit } \frac{CI}{IA} = \frac{CO}{OA}.$$

This is an equation which may be put into various forms.

IA is equal to CA - CI

and OA ,, ,, ,, CO - CA.

Hence the equation may be written—

$$\frac{CI}{CA-CI} = \frac{CO}{CO-CA}$$

$$\frac{CA}{CI} - 1 = 1 - \frac{CA}{CO}$$

$$\text{or } \frac{1}{CI} + \frac{1}{CO} = \frac{2}{CA}$$

This equation is in terms of distances measured from C, the centre of curvature of the mirror.

Also substituting for CI and CO, their values $\overline{CA - IA}$, and $\overline{OA + CA}$ respectively.

$$\frac{CA - IA}{IA} = \frac{OA + CA}{OA}$$

$$\text{Hence } \frac{CA}{IA} - \frac{CA}{OA} = 2$$

$$\text{or } \frac{1}{IA} - \frac{1}{OA} = \frac{2}{CA}$$

This equation is in terms of distances measured from A, the mirror, very often the most convenient form to adopt.

If, as is usual, we call u and v the distances from A of the object and image, measured positively in the direction from A to the object, and r the radius of the sphere measured from A in the same direction, this equation may be represented algebraically by the following:

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{r}$$

From this we may discuss the position of the image when that of the object is given, for any mirror whose radius of curvature is r, remembering that if the mirror is concave r is positive. It follows from general considerations, or from the symmetry of the above equation, that if the values a and b for u and v satisfy the above equation, it will also be satisfied for the values b and a for u and v. If therefore there are two points so connected, then the object being put in one of these points, the image will be found in the other.

Two points so connected are called a pair of conjugate points. There are an infinite number of such pairs of points for any mirror.

Refraction at a Spherical Surface.

Let O be the small object emitting light, and let C be the centre of a spherical surface A P upon which the light falls, separating the medium through which the light has come from another into which the light passes possessing a different index of refraction. Let μ be the velocity of propagation of light in the first medium relatively to its velocity in the second, *i.e.* the index of refraction. Let O P E be the path of a ray of light encountering the boundary in P. Join C P and produce it to K. Join C O, cutting the boundary in A and produce E P, backwards if necessary, to meet C O or C O produced in I.

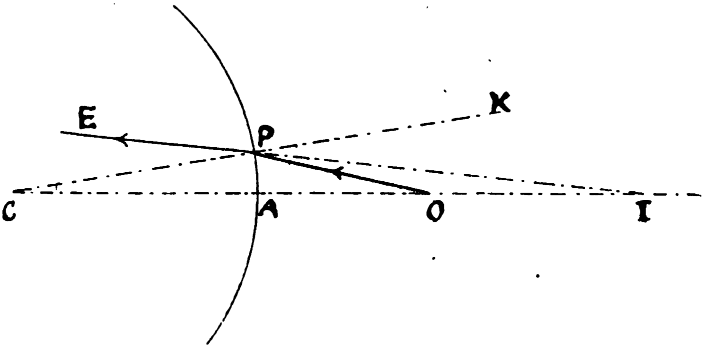

FIG. 9.

Then I will be the image of O for rays displaced from the position O P, by revolution round C O; but for other rays the image will not coincide with I unless P is very close to A. Vertical incidence is therefore necessary for a well-localized image. We must therefore find the ultimate position of I when P moves up to A.

In the triangle C P I,

$$\frac{\text{sine CPI}}{\text{sine PCI}} = \frac{\text{CI}}{\text{PI}}$$

$\therefore$ sine E P C which is equal to sine C P I

$$\text{is equal to } \frac{\text{CI}}{\text{PI}} \text{ sine PCI}$$

Also in the triangle C P O

$$\frac{\text{sine } CPO}{\text{sine } PCO}=\frac{CO}{PO}$$

∴ sine K P O which is equal to sine C P O

is equal to $\frac{CO}{PO}$ sine P C O

but by the laws of refraction

$$\text{sine } KPO=\mu \text{ sine } EPC$$

$$\therefore \frac{CO}{PO}=\mu\frac{CI}{PI}$$

and when P coincides with A this may be written

$$\frac{CO}{AO}=\mu\frac{CI}{AI}$$

$$\text{or } \frac{CO}{CI}=\mu\frac{AO}{AI}$$

This equation may be put into various forms as convenience may suggest.

If we wish to measure lines from the centre C of the sphere we may substitute C O − C A for A O

and C I − C A for A I

Whence the equation becomes

$$\frac{CO}{CI}=\mu\frac{CO-CA}{CI-CA}$$

$$\mu\, CI\; CA=\overline{\mu-1}\; CO\; CI+CA\; CO$$

$$\frac{\mu}{CO}=\frac{\overline{\mu-1}}{CA}+\frac{1}{CI}$$

If we wish to measure from A the point of incidence, we may write—

C A + A O for C O

C A + A I for C I

Whence the equation becomes

$$\frac{CA+AO}{CA+AI}=\mu\frac{AO}{AI}$$

$$\mathrm{AI}\ \mathrm{AC} = \overline{\mu - 1}\ \mathrm{AO}\ \mathrm{AI} + \mu\ \mathrm{AO}\ \mathrm{AC}$$

$$\text{or } \frac{1}{\mathrm{AO}} = \frac{\mu - 1}{\mathrm{AC}} + \frac{\mu}{\mathrm{AI}}$$

If as is usual we call u v r the distances from A to the object, the image, and the centre of the sphere, all measured positively in the direction opposite that in which the light is proceeding, we have $\frac{1}{u} - \frac{\mu}{v} = -\frac{\overline{\mu - 1}}{r}$

CHAPTER V

SIMPLE OPTICAL INSTRUMENTS

THESE may be enumerated and classified as follows :—

- (1) Mirrors
  - Plane
  - Spherical
    - Convex
    - Concave.
- (2) Plates.
- (3) Prisms.
- (4) Lenses
  - Convex
  - Concave.

They may be defined thus :—

Mirrors are polished surfaces employed as boundaries to reflect light incident upon them.

Plane mirrors are plane surfaces so polished.

Convex spherical mirrors are spherical boundaries whose convex sides are polished and receive the incident light.

Concave spherical mirrors are spherical boundaries whose concave or hollow sides are polished and receive the incident light.

Plates are masses of refracting material contained between two plane surfaces which are parallel.

Prisms are masses of refracting material contained between two plane surfaces which are inclined one to the other.

Lenses are masses of refracting material contained between two surfaces, either both spherical or one spherical and the other plane.

Convex lenses are often defined to be those which are thickest in the centre, *i. e.* in the region traversed by the line joining the centres of the two spheres, if the lens have two spherical surfaces, or in any other case by the line drawn from the centre of the one spherical surface at right angles to the plane surface.

Concave lenses are defined to be those which are thinnest in these regions.

We shall, however, in dealing with them be able to find a rather stricter definition than that given above for convex and concave lenses. However, if the lenses are thin, these definitions are sufficient and applicable.

It is clear that the simple instruments of the first group depend for their operation upon reflection, and that the remaining three groups depend upon refraction.

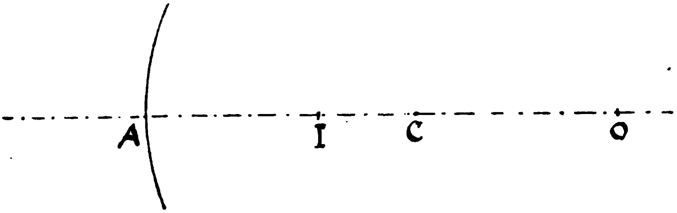

FIG. 10.

Plane mirrors. With regard to these instruments, probably enough has already been stated in dealing with a single plane reflecting surface. It may, however, be pointed out that the perfection of the images formed does not depend upon the vertical character of the incidence. Oblique pencils produce equally good images as those with vertical incidence. Hence plane mirrors are most admirably adapted to produce any required angular deflection of a pencil.

Spherical mirrors. We have also discussed the case of a pencil making vertical incidence upon a spherical reflecting surface, and have arrived at a formula:

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{r}$$

connecting the distance u of the object from the mirror with the distance v of the image from the same, the radius of the sphere being r. We give (Fig. 10) the diagram for the case of a concave mirror.

Here O is the position of the object $\therefore AO = u$
,, I ,, ,, image $\therefore AI = v$
,, C ,, ,, centre $\therefore AC = r$

We recall the fact, also previously proved, that

$$\frac{CI}{CO} = \frac{AI}{AO} = \frac{v}{u}$$

Magnification of the Image.

If the object, instead of being a mere point, is a small portion of a sphere at O, whose centre is at C, it is obvious that the image of each point will lie on a corresponding small portion of a sphere at I, whose centre is also at C. Each point in the object and its corresponding image will lie upon a straight line passing through C. Hence if a point in the object lie to one side of the line C O, the image of this point in the aggregate image will be on the other side of C O, and the relative distances from the line C O of these points in the object and image will be as C O to C I. Linear dimensions in the image will be therefore to the corresponding linear dimensions in the object as C I : C O.

But this ratio is also as $v : u$.

If we adopt the word magnification as expressing this ratio, and also call it negative when the image is inverted as regards the object, and express it by the symbol m, we arrive at the following mathematical measure of the magnification, viz.:

$$m = -\frac{v}{u}$$

If I lies on the positive side of A, the image is real and v is positive, but u being necessarily always positive, m in this case is negative. Hence a *real* image is *always inverted.*

Example.—A small object is placed 12 centimetres in front of a concave mirror, whose radius of curvature is 6 centimetres. Where will the image be situated and what will be its character?

Here $u = 12$, $r = 6$ $\therefore \frac{1}{u} + \frac{1}{v} = \frac{2}{r}$ becomes

$$\frac{1}{v} = \frac{2}{6} - \frac{1}{12} = \frac{1}{4} \text{ or } v = 4\text{; hence } m = -\frac{1}{3}$$

Hence the image will be formed at 4 centimetres from the mirror on the same side as the object; it will be real, inverted, and one-third the size of the object in linear dimensions.

From the equations

$$\left.\begin{aligned} \frac{1}{u} + \frac{1}{v} &= \frac{2}{r} \\ m &= -\frac{v}{u} \end{aligned}\right\}$$

we may eliminate either u or v, and obtain an equation only involving the other of these two and m and r.

For instance, supposing it desired to obtain an equation connecting the magnification with the position of the image, we multiply by v through the first equation and write $\frac{v}{u} = -m$

$$\text{Thus } -m + 1 = 2\frac{v}{r}$$

$$\text{or } v = \frac{r}{2}\left\{ 1 - m \right\}$$

$$\begin{aligned} \text{If } m &= -4 & v &= \frac{r}{2}5 \\ &= -3 & &= \frac{r}{2}4 \\ &= -2 & &= \frac{r}{2}3 \\ &= -1 & &= \frac{r}{2}2 \end{aligned}$$

$$\begin{aligned} \text{If } m &= 0 & v &= \frac{r}{2} \\ &= 1 & &= 0 \\ &= 2 & &= -\frac{r}{2} \\ &= 3 & &= -\frac{r}{2}2 \\ &= 4 & &= -\frac{r}{2}3 \end{aligned}$$

We readily see that any change in the magnification by unity involves a change in the position of the image by the distance $\frac{r}{2}$.

If the magnification decreases by unity algebraically (*e.g.* if m changes from -3 to -4), the image moves in the positive direction a distance $\frac{r}{2}$. This constant length is called the focal length of the mirror, and is frequently represented by one symbol f.

Hence $f=\frac{r}{2}$ and the equation between u and v is often written $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$

Again, multiply the former of the equations by u and substitute $-\frac{1}{m}$ for $\frac{u}{v}$

$$1-\frac{1}{m}=\frac{2}{r}u$$

$$\text{or } u=\frac{r}{2}\left\{1-\frac{1}{m}\right\}$$

In this case $\frac{1}{m}$ represents the size of the object relatively to the image, the inverse of the magnification as previously defined. Here it is again obvious that for every change in the value $\frac{1}{m}$ by unity, the object changes its position by $\frac{r}{2}$ or f.

$$
\begin{array}{llll}
\text{If } \dfrac{1}{m} = -2 & u = & \dfrac{r}{2} & 3 \\
\phantom{\text{If } \dfrac{1}{m}} = -1 & = & \dfrac{r}{2} & 2 \\
\phantom{\text{If } \dfrac{1}{m}} = 0 & = & \dfrac{r}{2} & \\
\phantom{\text{If } \dfrac{1}{m}} = 1 & = & & 0 \\
\phantom{\text{If } \dfrac{1}{m}} = 2 & = & -\dfrac{r}{2} & \\
\phantom{\text{If } \dfrac{1}{m}} = 3 & = & -\dfrac{r}{2} & 2
\end{array}
$$

Hence we may make the following diagram for a concave mirror:

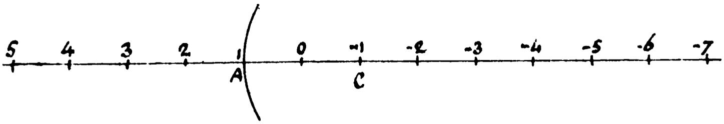

Fig. 11.

Let A be the position of the mirror, C its centre of curvature. Join C with A the centre of the mirror itself, and produce A C both ways indefinitely. Bisect A C in a point against which write the cipher 0. Against A write 1 and at C write -1, and mark off successive distances all equal to half the radius of curvature, numbering them as above in order; and subdividing the scale so formed to any desired extent.

This simple diagram has the following properties.

If an object is put down at any position, say -4, the image will be found at the point indicated by inverting this figure, *i. e.* at $-\frac{1}{4}$, or at one quarter of the way from 0 to -1. The size of the image relatively to the object will be $(-\frac{1}{4})$.

It is clear that if the object be placed at point $(-\frac{1}{4})$ the image will be found at the point (-4).

In fact the point represented by any number (a) is conjugate to the point $\left(\frac{1}{a}\right)$.

The point (0) is therefore conjugate to a point at infinity $\left(\frac{1}{o}\right)$. It is therefore the point at which we should find the image of the sun or any other object at a practically infinite distance. From its supposed importance it has been called the *principal focus* of the mirror. By the Germans it is called Brennpunkt, which may be translated Burn-point, from the fact that, rays of heat being subject to the same laws of reflection as light, bodies may be ignited when placed at this point, the mirror being at the same time exposed, at vertical incidence, to the sun's rays.

The points which bear a positive number involve no difficulty.

If an object is placed between 0 and 1, say at $\frac{1}{3}$, its image will be at the point 3, and it will not be inverted on account of the absence of the negative sign. It will of course be virtual, and three times the size of the object.

It will be proper here to utter a caution as to the use of the term magnification, to which the symbol m has been adjudicated. It refers only to the relation of image to object in the case of lines drawn at right angles to the direction of the pencil of light. Suppose the object to be three small straight lines mutually bisecting each other at right angles. Let these be a, b, and c, of which a lies in the direction of the axis of the pencil of light, and the other two, b and c, at right angles to this direction. If m is the magnification as previously defined, the two axes b and c will have for image $m\,b$ and $m\,c$, in accordance with what has been stated. But a will have for its image a line equal to $-m^2 a$, as may be easily proved by forming the value of $\frac{dv}{du}$ from the equation $\frac{1}{u}+\frac{1}{v}=\frac{2}{r}$, which will be found to be

$$-\frac{v^2}{u^2} \text{ which is } -m^2,$$

the form showing that the quantity is necessarily a negative one. End inversion, therefore, must always take place in such an image.

The volume of the small parallelopiped, whose edges are respectively a, b, and c, is abc, but the volume of the image will be $-m^2\,a \times m\,b \times m\,c = -m^4\,abc$, a necessarily negative quantity. The surface ab of the object will have for its image a surface equal to $-m^2\,a \times mb$

$$= -m^3\,ab.$$

Similarly for $a\,c$ the image will be $-m^3\,a\,c$, but for the surface $b\,c$ the image will be $m^2\,b\,c$, which has a positive value whatever value m may have.

A volume image is accordingly necessarily reversed and distorted, and that of a surface at right angles to the direction of light propagation necessarily correct. But the image of a plane surface having extension in the direction in which light is travelling will be distorted but not reversed when m is negative, *i.e.* in the case of a real image.

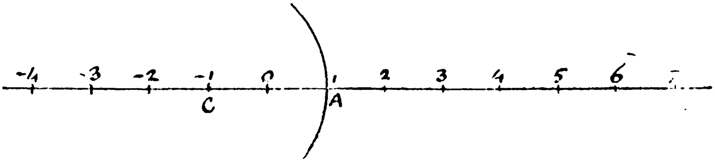

FIG. 12.

A small sphere would have an oblate or prolate spheroid for its image according as m is less or greater than unity arithmetically.

In the cases of m being equal to $+1$ or -1 alone will there be no distortion from the spherical shape.

Hence the image of a small sphere hard against the mirror is a sphere.

The above diagram refers to a convex mirror. The light is supposed as before to come from the right-hand side, and the letters and figures have the same significance as before.

It will be noticed that this is the same diagram as before, reversed right and left. But the convexity and concavity are also reversed in the same way.

Point 1 is always against the mirror, and Point − 1 is always at the centre of curvature.

Passage of Light through a Plate.

In discussing the course of a beam of light subject to refraction first at one plane surface and secondly at another parallel to the first, it has been shown that the direction of the ray in the third medium is independent of the index of the second. If therefore the third medium is identical with the first, the ray in it will be parallel to the direction in the first; for we may imagine that the second medium is also identical with the first, in which case there can be no angular displacement.

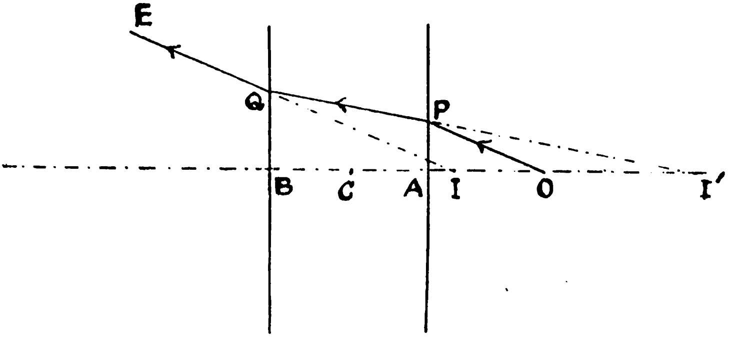

FIG. 13

A plate therefore shifts a ray laterally, but gives it no deviation in angle.

Let O P Q E be a ray from an object O, passing through a plate contained between the parallel planes A P, B Q, and let O A B be a perpendicular from O upon the two planes. Let $AB = d$, and let μ be the index of the material of the plate relatively to medium on either side.

P Q is the path of the ray within the plate. Produce Q P, E Q to meet the line B A O, in I′ and I respectively.

Then the ultimate positions of I′ and I when P moves up to A, will be the positions of the images formed by the first and second refraction respectively. Hence—

$$\therefore\ \mathrm{AI'} = \mu\ \mathrm{AO},$$

$$\text{and } \mathrm{BI} = \frac{1}{\mu}\mathrm{BI'}.$$

If we refer all measurements to a point C bisecting A B, we must write—

$$\mathrm{AI'} = \mathrm{CI'} - \frac{d}{2}$$

$$\mathrm{AO} = \mathrm{CO} - \frac{d}{2}$$

$$\mathrm{BI} = \mathrm{CI} + \frac{d}{2}$$

$$\mathrm{BI'} = \mathrm{CI'} + \frac{d}{2}$$

Whence the above equations become

$$\mathrm{CI'} - \frac{d}{2} = \mu\left\{\mathrm{CO} - \frac{d}{2}\right\}$$

$$\mathrm{CI} + \frac{d}{2} = \frac{1}{\mu}\left\{\mathrm{CI'} + \frac{d}{2}\right\}$$

Whence eliminating C I′ we have

$$\overline{\mathrm{CO} - \mathrm{CI}} = \frac{\overline{\mu - 1}}{\mu}d.$$

The effect therefore of a plate, when the incidence is vertical, is to produce an image nearer to or farther from the plate than the object is according as the value of μ is greater or less than unity, by a distance $\frac{\mu - 1}{\mu}d$ which does not depend upon the distance C O. This will therefore be true if the object moves up to coincidence with A.

The flecks on the surface of a plate seen through the plate thus always appear nearer to the eye than they really are by the distance

$$\frac{\mu - 1}{\mu} d$$

when d is the thickness of the plate.

The Passage of a Ray through a Prism.

A ray may fall upon one of the faces of a prism in a variety of directions and with a variety of angles of incidence, but we shall here only consider those cases in which the ray passes in what is called a principal plane, *i.e.* in a plane which cuts at right angles the edge formed by the intersection of the two faces of the prism.

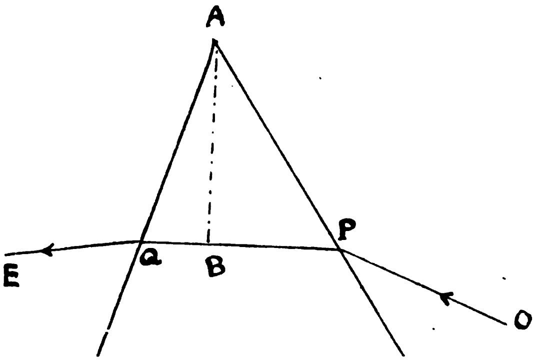

FIG. 14.

Let A P Q be a principal plane of the prism whose edge is represented by a straight line drawn through the point A at right angles to this plane. Then if Q and P are points in the principal plane, the angle P A Q is that at which the two faces are inclined one to the other, and is called the refracting angle of the prism.

Let O P Q E be a ray passing through the prism, and obeying at the points of incidence P and Q the laws of refraction. If a line A B is drawn in the plane of the ray from the refracting angle, at right angles to the path of the ray within the prism, this line will either bisect the refracting angle or it will divide it into two unequal parts.

In either case we may consider the prism as divided into two prisms whose refractive angles make up that of the single prism, one face of one prism coinciding with one of the other.

We shall then simply have to deal with two cases of a ray passing through a prism the incidence at one side being vertical.

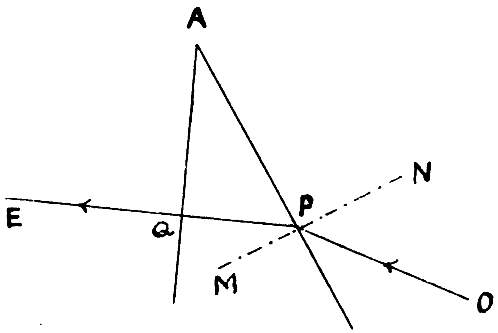

FIG. 15.

Let O P be a ray incident at P upon one side of a prism and passing into the prism, whose refracting angle is α and incident vertically at Q upon the other side, and passing out without further bending to E.

Let μ be the index of the material of which the prism is made, and let d be the deviation of the ray at P.

Draw M P N a normal at P to the face upon which O P is incident.

O P N is the angle of incidence, and Q P M that of refraction.

The latter is equal to the angle a, the refracting angle of the prism, for they are each equal to

$$\left(\frac{\pi}{2} - \text{APQ}\right).$$

Calling OPN ϕ we have therefore

$$\sin \phi = \mu \sin a.$$

The deviation d of the ray at P is equal to the

$$\text{OPN} - \text{QPM}$$

or
$$d = \phi - a$$

From these two equations we may eliminate ϕ and obtain

$$\sin (d + a) = \mu \sin a$$

an equation enabling us to determine d for a prism whose refracting angle is a, incidence being vertical on one of the faces.

If β be the refracting angle of a second prism whose face coincides with AQ, or is parallel with it, and therefore receives the ray QE at vertical incidence, the deviation d' will be given by the equation

$$\sin (d' + \beta) = \mu \sin \beta$$

and the angle of final emergence ϕ' may be found from

$$d = \phi' - \beta,$$
$$\text{or } \phi' = (d' + \beta),$$

the total deviation being $d' + d$, and the refracting angle of the prism made up of the two parts $(a + \beta)$.

EXAMPLE.—Suppose the refracting angle of the prism to be 60° and the angle of incidence 55°, and that $\mu = \frac{3}{2}$.

To find the separate and aggregate deviations and the angle of emergence.

Then the angle of refraction a will be determined by

$$\sin a = \frac{\sin 55^\circ}{\frac{3}{2}}, \text{ whence } a = 33^\circ\ 6'\ 0''$$

$$\text{and } d = \phi - a = 21^\circ\ 54'\ 0''$$

Therefore the angle β will be $60^\circ - 33^\circ\ 6' = 26^\circ\ 54'\ 0''$

$$\sin \phi' = \sin (d' + \beta) = \frac{3}{2} \sin 26^\circ\ 54'$$

$$\therefore (d' + \beta) = \phi' = 42^\circ\ 44'\ 18''$$

$$\text{and } d' = 15^\circ\ 50'\ 18''$$

$$\text{The total deviation} = d + d' = 37^\circ\ 44'\ 18''$$

If the ray passes symmetrically through the prism, then $a = \beta$, each of these angles being half the refracting angle of the prism.

In this case we have merely to calculate the deviation at one incidence, and multiply by 2.

The angle of incidence must then accord with the equation $\sin \phi = \mu \sin a$.

Example.—The refracting angle is 60° and $\mu = \frac{3}{2}$.

What must the incidence be for the symmetrical passage of the ray, and what will be the total deviation?

$$\text{Here } a = 30^\circ\ 0'\ 0''$$

and $\sin \phi = \frac{3}{2} \sin 30 = \frac{3}{4}$ $\qquad \therefore \phi = 48^\circ\ 35'\ 25''$

The deviation at one refraction $= \phi - a = 18^\circ\ 35'\ 25''$ and the total deviation, being double this, is $37^\circ\ 10'\ 50''$.

Observe that the deviation obtained in the former example was somewhat greater than in this case. The rule is a general one. In the symmetrical position the deviation is a *minimum*, in all cases, but when μ is less than unity the deviation is always towards the edge of the prism, whereas when μ is greater than unity (glass prisms in air) the deviation is always away from the edge.

Approximate formulæ for prisms of small refracting angle.

When the incidence is nearly vertical upon the face of a small-angled prism, all the other angles, viz. that of the first refraction, those of the first and second incidence, and that of emergence, will also be very small, so that we may for the sines of these angles employ their circular measure.

In this case

$$\phi = \mu\,\alpha$$
$$\phi' = \mu\,\beta$$

The whole deviation $\Delta = \phi - \alpha + \phi' - \beta$

$\alpha + \beta = i$ the refracting angle of the prism.

Hence $\Delta = \overline{\mu - 1}\,\overline{\alpha + \beta}$

$= \overline{\mu - 1}\,i,$

or the deviation is independent of the actual incidence,

CHAPTER VI

LENSES. PRELIMINARY CONSIDERATIONS

In discussing the simple case of one spherical refracting surface, we have arrived at the formula :—

$$\frac{1}{u} - \frac{\mu}{v} = -\frac{\mu - 1}{r}$$

It must be remembered that u is necessarily positive, and is the distance of the object from the surface.

The radius of the spherical surface, r, is to be considered positive if the surface is concave, for then A C is measured in the same direction as A O.

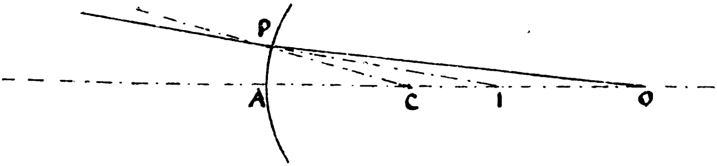

FIG. 16.

The dependent variable, v, in this equation, if positive, must be represented also by a measurement from A, as A I, made in the same direction.

We have also established the fundamental relation :—

$$\frac{\text{C O}}{\text{C I}} = \mu\,\frac{\text{A O}}{\text{A I}}$$

Now if the object, instead of being a mere point, is a small portion of a sphere at O, whose centre is C, it is obvious that the image of each point will lie on a corresponding small portion of a sphere at I, whose centre is also at C. Each point in the object and its corresponding image will lie upon a line passing through C.

Hence if a point in the object lie to one side of the line C O, the image of this point will be on the same side of C O, and the relative distances from the line C O of these points will be as C O to C I.

Linear dimensions in the image will therefore bear to the corresponding linear dimensions in the object the relation C I : C O.

If we adopt the word magnification as expressing this ratio, and call it positive when corresponding points are on the same side of C O, *i.e.* when the image is erect, and give it the symbol m, then

$$m=\frac{\mathrm{CI}}{\mathrm{CO}}$$

But the ratio $\frac{\mathrm{CI}}{\mathrm{CO}}$ is equal to $\frac{1}{\mu}\frac{\mathrm{AI}}{\mathrm{AO}}$

Therefore also

$$m=\frac{1}{\mu}\frac{\mathrm{AI}}{\mathrm{AO}}=\frac{1}{\mu}\frac{v}{u}$$

From this and from the equation

$$\frac{1}{u}-\frac{\mu}{v}=-\frac{\overline{\mu-1}}{r}$$

we may eliminate u or v at pleasure and obtain an equation connecting m and v, or m and u.

Thus we have

$$\left.\begin{aligned} m&=1-\frac{\overline{\mu-1}}{\mu r}v \text{ which is independent of } u \\ \frac{1}{m}&=1+\frac{\overline{\mu-1}}{r}u \quad ,, \quad ,, \qquad ,, \qquad ,, \; v \end{aligned}\right\}$$

Taking the first of these equations we see the truth of the following statements :—

If $m=0 \qquad v=\frac{\mu r}{\mu-1}$
$m=1 \qquad v=0$
$m=2 \qquad v=-\frac{\mu r}{\mu-1}$
$m=3 \qquad v=-\frac{2\mu r}{\mu-1}$
$m=4 \qquad v=-\frac{3\mu r}{\mu-1}$
$m=-1 \qquad v=\frac{2\mu r}{\mu-1}$
$m=-2 \qquad v=\frac{3\mu r}{\mu-1}$

Taking the second equation and writing n for $\frac{1}{m}$, the signification of n being the linear relation of object to image,

If $n=0 \qquad u=-\frac{r}{\mu-1}$
$n=1 \qquad u=0$
$n=2 \qquad u=\frac{r}{\mu-1}$
$n=3 \qquad u=\frac{2r}{\mu-1}$
$n=4 \qquad u=\frac{3r}{\mu-1}$
$n=-1 \qquad u=-\frac{2r}{\mu-1}$
$n=-2 \qquad u=-\frac{3r}{\mu-1}$

The common difference in v for a change of m by unity is $-\frac{\mu r}{\mu-1}$

The common difference in u for a change of n by unity is $\frac{r}{\mu-1}$

In drawing a diagram analogous to that already explained in the case of spherical mirrors, we shall want two scales. We shall make that corresponding to u *above* the axis of the surface, and that corresponding to v *below* the axis.

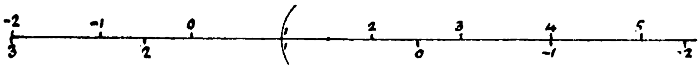

FIG. 17.

The common point for the numbers is against the surface where both scales have the number 1.

The distance between the point 2 on the upper scale and the point 0 on the lower scale is (numerically)

$$\frac{\mu r}{\mu - 1} - \frac{r}{\mu - 1} = r \text{ the radius of the spherical surface.}$$

Also the ratio of the two distances 1 to 0 on the lower scale, to 1 to 0 on the upper scale, is μ.

If the object is at infinity the image is at the point 0 of the lower scale, and is virtual. As the object moves up to the surface the image does so too, is always erect, and virtual.

In the above diagram r is considered to be positive, and μ greater than unity. According as the conditions change we may have four distinct cases.

$\mu > 1$ r positive.
$\mu > 1$ r negative.
$\mu < 1$ r positive.
$\mu < 1$ r negative.

We now give diagrams for the three latter cases.

$\mu > 1$ r negative.

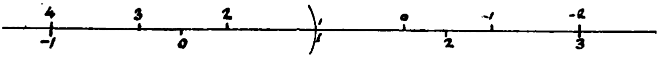

FIG. 18.

$\mu < 1$ r positive.

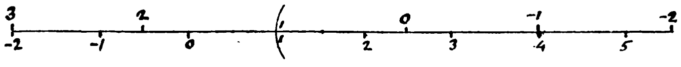

FIG. 19.

$\mu < 1$ r negative.

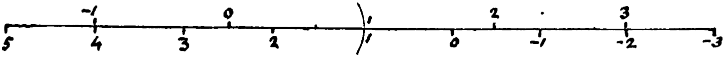

FIG. 20.

Of these four diagrams the fourth is the same as the first turned upside down, and the second and third are connected in the same way.

In these cases there is not the conjugate relation between pairs of points which has been noticed in the case of spherical mirrors.

For instance, taking the first of the four diagrams, suppose an object is placed at the point marked 3 in the upper scale. This means that the object will be three times the linear scale of the image. Hence the image being $\frac{1}{3}$ of the size of the object, it will be found at the point $\frac{1}{3}$ of the lower scale. As drawn the diagrams refer to the case of $\mu=\frac{3}{2}$, and this point will be therefore coincident with that marked 2 in the upper scale. But if we place an object at this point 2 the image will have to be at the point $\frac{1}{2}$ of the lower scale, which does not coincide with the point 3 of the upper scale where we originally supposed the object placed. Nor can we say that there is one focal length for the system, using that expression for the distance between any two points on the same scale which differ in denomination by unity.

We might perhaps say that there is one focal length for the light in one medium, and another for the light in the second medium, and that they are related to one another in the ratio $1 : -\mu$.

But the scales invariably run in opposite directions, and have their points marked 1 in coincidence with the surface and with one another.

As the relative size of the object to the image varies inversely as the size of the image to the object, and the scales run in different directions, the image and object always move in the same direction in space.

Lenses being portions of a refracting material contained between two spherical surfaces, it is clear that they could be discussed by superimposing the diagrams proper to the two surfaces, putting the points marked 1 at that distance apart which is equal to the thickness of the lens at its axis.

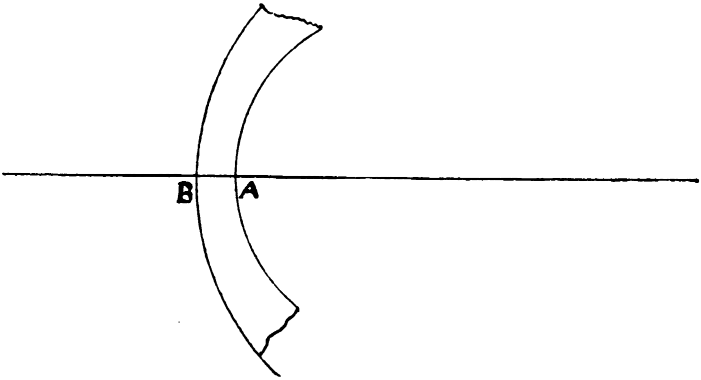

Fig. 21.

We can gain the same end by the use of the equations appropriate to the two refractions.

Let A be the position of one surface, and B of the other, and let $AB = d$. Let r_1 and r_2 be the two radii of curvature, and let u be the distance from A of the object, and v be the distance from B of the final image.

Let v' represent the distance of the image by the first refraction from A.

And u' be the distance of that image from B, so that $u'=v'+d$.

Let $m_1\, m_2$ be the magnifications due to the two refractions taken separately.

And let m be the total magnification, which is necessarily equal to the product $m_1\, m_2$.

$$\text{Now } m_1=\frac{v'}{\mu u} \text{ and } m_2=\frac{\mu v}{u'}$$

$$\therefore\ m=\frac{v\,v'}{u\,u'}$$

We also have the two equations,

$$\frac{1}{u}-\frac{\mu}{v'}=-\frac{\mu-1}{r_1}$$

$$\frac{1}{u'}-\frac{1}{\mu v}=-\frac{\frac{1}{\mu}-1}{r_2}$$

Between the variable quantities we have therefore the following equations:

$$\left.\begin{array}{lr}\dfrac{1}{u}-\dfrac{\mu}{v'}=-\dfrac{\mu-1}{r_1} \quad . \quad . \quad . \quad . \quad . & (1)\\[2ex] \dfrac{1}{u'}-\dfrac{1}{\mu v}=-\dfrac{\frac{1}{\mu}-1}{r_2} \quad . \quad . \quad . \quad . \quad . & (2)\\[2ex] v'+d=u' \quad . \quad . \quad . \quad . \quad . \quad . & (3)\\[2ex] \dfrac{v\,v'}{u\,u'}=m \quad . \quad . \quad . \quad . \quad . \quad . & (4)\end{array}\right\}\ \text{A}$$

From these equations it is evident that we can eliminate $u'\ v'$ and either u or v, and obtain an equation between m and either u or v, involving otherwise only the fixed dimensions of the lens, viz. $r_1\, r_2$ and d.

It will, however, be found convenient to introduce two other dimensions of importance, defined and symbolized as follows:—

Let u_0 be the value of u for an infinite value of v, and
v_0 ,, ,, ,, ,, v ,, ,, ,, ,, ,, u.

To find these values:

Put $v = \text{infinity}$ in the equation (2).

$$\text{Then } u' = \frac{\mu r_2}{\mu - 1}$$

$$\therefore \text{ from equation (3) } v' = \frac{\mu r_2}{\mu - 1} - d = \mu\left\{\frac{r_2 - \frac{\mu - 1}{\mu} d}{\mu - 1}\right\}$$

Hence from equation (1) u will equal u_0 when v' has this value.

$$\text{Hence } \frac{1}{u_0} = \frac{\mu - 1}{r_2 - \frac{\mu - 1}{\mu} d} - \frac{\mu - 1}{r_1}$$

$$= \overline{\mu - 1}\,\frac{\left\{r_1 - r_2 + \frac{\mu - 1}{\mu} d\right\}}{r_1\left(r_2 - \frac{\mu - 1}{\mu} d\right)}$$

If for the sake of abbreviation we write

$$s = \overline{\mu - 1}\,\{\mu(r_2 - r_1) - \overline{\mu - 1}\,d\}$$

$$\text{and } f = \frac{\mu r_1 r_2}{s}$$

which, it will be noticed, are merely functions of the lens dimensions and index,

$$\text{then } u_0 = r_1 \frac{\overline{\mu - 1}\,d}{s} - f$$

Proceeding in a similar way to obtain v_0 we find

$$v_0 = r_2 \frac{\overline{\mu - 1}\,d}{s} + f.$$

f is called the focal length of the lens, and upon its value and sign will be found to depend most of the important properties of a lens. For the moment it will be treated merely as a convenient abbreviation.

s is also a very important quantity, as its sign affects that of f. It has, however, not received any particular name. Both f and s are of one dimension in space.

We now proceed to the elimination of u' and v' in equations (1) and (2 and 3) of Group A, p. 49.

From equation (1)

$$v' = \frac{\mu r_1 u}{r_1 + \overline{\mu - 1}\, u}$$

and from equation (2)

$$u' = \frac{\mu r_2 v}{r_2 + \overline{\mu - 1}\, v}$$

Since from equation (3)

$$d = u' - v'$$

$$d = \frac{\mu r_2 v}{r_2 + \overline{\mu - 1}v} - \frac{\mu r_1 u}{r_1 + \overline{\mu - 1}u}$$

an equation only involving u and v as variable with the position of the object.

Multiplying up and arranging terms, the above becomes

$$\left.\begin{array}{l} uv.\overline{\mu - 1}\,\{\overline{\mu - 1}d - \mu.\overline{r_2 - r_1}\} \\ + u\,\{r_2.\overline{\mu - 1}d + \mu r_1 r_2\} \\ + v\,\{r_1\,\overline{\mu - 1}d - \mu r_1 r_2\} \\ + d r_1 r_2 \end{array}\right\} = 0$$

or taking advantage of the abbreviations f and s and of the expressions for u_0 and v_0,

$$uv - uv_0 - vu_0 - \frac{df}{\mu} = 0$$

$$\text{or } \overline{u - u_0}\ \overline{v - v_0} = u_0 v_0 + \frac{df}{\mu}$$

By multiplying the expressions for u_0 and v_0 together, we easily obtain the equation

$$u_0 v_0 + \frac{df}{\mu} = -f^2$$

$$\therefore\ \overline{u - u_0}\ \overline{v - v_0} = -f^2$$

If we agree to measure the distances of the object and image from the points indicated by $u = u_0$ and $v = v_0$

respectively, and indicate these distances by U and V respectively, we have finally

$$\mathrm{U}\,\mathrm{V} = -f^2$$

The points at distance u_0 from the first surface and v_0 from the second surface respectively are called the principal foci of the lens.

If we measure from the first of these points to the object, this distance is represented by U, and if we measure from the second of these points to the image, this distance is represented by V, and U and V are connected by the equation

$$\mathrm{U}\,\mathrm{V} = -f^2$$

when $f = \frac{\mu r_1 r_2}{s}$

and $s = \overline{\mu - 1}\ \{\mu \overline{r_2 - r_1} - d\,\overline{\mu - 1}\}$

The position of the principal foci are given by the equations

$$u_0 = \frac{r_1}{s}\left(\overline{\mu - 1} - \mu r_2\right)$$

$$v_0 = \frac{r_2}{s}\left(\overline{\mu - 1} + \mu r_1\right)$$

It remains to express the magnification m in terms of V, U, and f.

$$m = \frac{vv'}{uu'}$$

Now $\frac{v'}{u} = \frac{\mu r_1}{r_1 + \overline{u - 1}\ u}$

and $\frac{u'}{v} = -\frac{\mu r_2}{r_2 + \overline{\mu - 1}\ v}$

Hence $m = \frac{r_1}{r_2}\left(\frac{r_2 + \overline{\mu - 1}\ v}{r_1 + \mu - 1\ u}\right)$

For u write $\mathrm{U} + u_0$ or $\mathrm{U} + \frac{r_1}{s}\ (\overline{\mu - 1}\ d - \mu r_2)$

and for v ,, $\mathrm{V} + v_0$ or $\mathrm{V} + \frac{r_2}{s}\ (\overline{\mu - 1}\ d + \mu r_1)$

Hence $r_2 + \overline{\mu - 1}\, v = \overline{\mu - 1} \left\{ \mathrm{V} + \frac{r_2^2}{s} \mu \right\}$

Similarly $r_1 + \overline{\mu - 1}\, u = \overline{\mu - 1} \left\{ \mathrm{U} - \frac{r_1^2}{s} \mu \right\}$

Hence

$$m = \frac{r_1}{r_2} \frac{\mathrm{V} + r_2^2 \frac{\mu}{s}}{\mathrm{U} - \frac{r_1^2 \mu}{s}}$$

$$= \frac{\mathrm{V} \left\{ \frac{r_1}{r_2} + \frac{r_1 r_2 \mu}{s \mathrm{V}} \right\}}{\mathrm{U} - \frac{r_1^2 \mu}{s}}$$

Now $\mathrm{U} = -\frac{f^2}{\mathrm{V}}$ and $f = \frac{r_1 r_2 \mu}{s}$

$$\therefore m = \frac{\mathrm{V} \left\{ \frac{r_1}{r_2} + \frac{f}{\mathrm{V}} \right\}}{-f \left\{ \frac{f}{\mathrm{V}} + \frac{r_1}{r_2} \right\}}$$

$$= -\frac{\mathrm{V}}{f}.$$

$$\therefore \mathrm{V} = -fm.$$

and $\mathrm{U} = \frac{f}{m}$

Whence $\frac{\mathrm{V}}{\mathrm{U}} = -m^2$

We thus see that with any lens there are two points called the principal foci, one corresponding to the light before the encounter with the lens, the other to the emergent light, of such a character that if incident light passes through the first of these points, at emergence it will be parallel to the axis of the lens, and that if parallel to the axis of the lens while incident, on emergence it will pass through the second principal focus.

The magnification of the image is proportional to its distance from the second principal focus, and the size of the object relative to the image is proportional to the object's distance from the first principal focus.

Magnification is changed by unity for a motion in either direction of the image, equal to the constant quantity *f*, called the focal length and symbolized by the letter *f*.

The lens may have a positive value for *f*, or a negative value.

If *f* has a negative value, the lens is called convex; if a positive value, concave.

f may have all possible values, including infinity and zero.

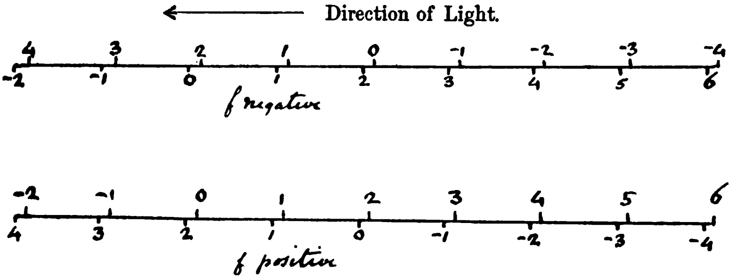

FIG. 22.

We shall, therefore, devote some space to discussing the values of the focal length.

Take any line to represent the axis of the lens and let the principal foci be at the points o o, the first marked off above the axis, the second below. From these points set off distances equal to *f* backwards and forwards, above and below.

Let the points so obtained be numbered in order as shown: the top scale of the top diagram and the bottom scale of the bottom diagram having the numbers increasing in the direction of the light, the two other scales running in the opposite direction. Then the top diagram refers to a convex lens where *f* is negative, and the lower diagram to a concave lens where *f* is positive.

The truth of this is conveyed in the equations

$$\text{v} = -fm \qquad \text{u} = \frac{f}{m}$$

As in similar diagrams obtained for mirrors, if the object is situated at any point whose number is n of the upper scale, the image will be found at the point whose number is $\left(\frac{1}{n}\right)$ of the lower scale, and the size of the image will be $\left(\frac{1}{n}\right)$ of the size of the object.

From the equations

$$f = \frac{\mu r_1 r_2}{\text{s}}$$

$$s = \overline{\mu - 1}\,\{\mu(r_2 - r_1) - \overline{\mu - 1}\,d\}$$

we can discuss the value and sign of f for any particular case, and from the equations

$$u_0 = r_1 \frac{\overline{\mu - 1}\,d}{s} - f$$

$$v_0 = r_2 \frac{\overline{\mu - 1}\,d}{s} + f$$

we can determine the positions of the principal foci relatively to the lens itself.

Ordinary double convex lens of equal numerical radii of curvature, and thin.

$$\text{Here } r_1 = -r_2 \qquad \therefore f = -\frac{\mu r_2^2}{s}$$

$$\text{and } s = \overline{\mu - 1}\,\{2\mu \text{r}_2 - \overline{\mu - 1}\,d\}$$

If we have $\mu > 1$, and $2\mu r_2 > \overline{\mu - 1}\,d$,

Then s is positive and f necessarily negative.

This typical case, when the surfaces are convex in the proper sense of the word, has caused lenses which have a negative value for their focal length to be called convex lenses.

Ordinary double concave lens, of equal numerical radii of curvature, and thin.

$$\text{Here } r_2 = -r_1, \text{ therefore } f = -\frac{\mu r_1^2}{s}$$

$$\text{and } s = -\overline{\mu - 1}\,\{2\mu r_1 + \overline{\mu - 1}\,d\}$$

and if $\mu > 1$, then s is necessarily negative and therefore f is positive. This case, in which the surfaces are concave, has led to the word being applied to lenses having a positive focal length. That the use of these words concave and convex to lenses is not entirely logical may be seen by the fact that a double convex lens of equal curvatures, if sufficiently thick, will have a positive value for its focal length with the general properties of a concave lens.

In such a case, to make f positive, s must be negative, and this is the case where

$$\overline{\mu - 1}\,d > 2\mu r_2$$

$$\text{or } d > 2\frac{\mu}{\mu - 1}r_2$$

Suppose, for instance, that $\mu = 1{\cdot}5$.

Then $d > 6r_2$ is the condition for a positive focal length.

A long rod of glass rounded similarly at each end may therefore very possibly possess the general characters of a concave lens.

The distance between the principal foci (D) is given by the equation

$$\begin{aligned} \mathrm{D} &= v_0 - (u_0 + d) \\ &= 2f + \overline{\mu - 1}\,\frac{d}{s}\left(r_2 - r_1\right) - d \\ &= 2f + \frac{d}{s}\left(\overline{\mu - 1}\;\overline{r_2 - r_1} - s\right) \end{aligned}$$

Hence if $\mathrm{D} = 2f$

$$s = \overline{\mu - 1}\;\overline{r_2 - r_1}$$

$$\mu\,(r_2 - r_1) - \overline{\mu - 1}\,d = \overline{r_2 - r_1}$$

$$\text{or } \overline{r_2 - r_1} = d$$

Hence the surfaces must be concentric. A sphere is a case. Two hemispheres of unequal radii placed with their flat sides together, and concentric, are another case. A spherical shell is a third.

From the values of u_0 and v_0 already given we may eliminate the auxiliary quantity s, by using its value in the equation $f = \frac{\mu r_1 r_2}{s}$, and so obtain :—

$$u_0 = f\left\{\frac{\mu - 1}{\mu}\frac{d}{r_2} - 1\right\}$$

$$v_0 = f\left\{\frac{\mu - 1}{\mu}\frac{d}{r_1} + 1\right\}$$

If these are to be equal, but of opposite signs

$$\frac{\mu - 1}{\mu}\frac{d}{r_2} - 1 = -\frac{\mu - 1}{\mu}\frac{d}{r_1} - 1$$

$$\therefore \frac{1}{r_2} = -\frac{1}{r_1} \text{ or } r_1 = -r_2$$

A symmetrical lens, therefore, is the only sort which will fulfil this condition.

If u_0 is equal to f, then—

$$\frac{\mu - 1}{\mu}\frac{d}{r_2} - 1 = 1$$

$$\frac{\mu - 1}{\mu}\frac{d}{r_2} = 2$$

$$r_2 = \frac{\mu - 1}{2\mu} d, \text{ or } d = \frac{2\mu}{\mu - 1} r_2$$

In this case r_2 must be positive.

If $u_0 = -f$, then $\frac{\mu - 1}{\mu}\frac{d}{r_2} - 1 = -1$

$\therefore d = 0$

an ideal condition, impossible of fulfilment, or r_2 is infinite.

Lenses having their second face plane fulfil this condition.

CHAPTER VII

COMBINATIONS OF LENSES, FORMULÆ, TELESCOPE, MICROSCOPE

THE extremely simple mathematical forms which have been shown to be true in a simple lens will be found to be true for two or any number of lenses having a common axis. This we proceed to show for two lenses, the proof in that case being capable of the simplest generalization so as to be available for the more general one.

$$\text{Let } u_1 v_1 = -f_1^2$$
$$v_1 = -f_1 m_1$$
$$u_1 = \frac{f_1}{m_1}$$

be the equations applicable to one lens, and let similar equations with subscripts 2 be applicable to a second lens. Also let the distance from the second principal focus of the first lens be at a distance k in front of the first principal focus of the second lens: then

$$v_1 + k = u_2$$

and if m be the final magnification

$$m = m_1 m_2$$

Eliminating v_1 and u_2 we have

$$k = \frac{-f_2^2}{v_2} + \frac{f_1^2}{u_1}$$

or
$$\frac{f_1^2}{k} \frac{1}{u_1} - \frac{f_2^2}{k v_2} = 1$$

From this equation we see that if v_2 is infinite u_1 is equal to $\frac{f_1^2}{k}$.

Call this value u_0.

The point indicated is a principal focus of the combination.

Similarly if u_1 is infinite

$$v_2 \text{ is equal to } -\frac{f_2^2}{k}$$

Call this value v_0. It gives the position of the second principal focus.

Hence the equation may be written

$$\frac{u_0}{u_1} + \frac{v_0}{v_2} = 1$$

$$\text{or } u_1v_2 - u_1v_0 - u_2v_0 = 0$$

$$\text{or } (u_1 - u_0)(v_2 - v_0) = u_0v_0$$

$$= \frac{f_1^2 f_2^2}{k^2}$$

If $u = u_1 - u_0$

$v = v_2 - v_0$

$$\text{We have } uv = -\left(\frac{f_1f_2}{K^2}\right)^2$$

$$\text{and } m = m_1m_2 = -\frac{f_1}{f_2} \cdot \frac{v_2}{u_1} = -\frac{f_1}{f_2} \cdot \left(\frac{v_2}{u_0} - \frac{v_0}{u_0}\right)$$

$$= -\frac{v_2 - v_0}{\left(\frac{f_1f_2}{k}\right)} = -\frac{v}{\left(\frac{f_1f_2}{k}\right)}$$

$$\text{or } = \left(\frac{f_1f_2}{k}\right)\frac{1}{u}$$

$$\therefore u = \left(\frac{f_1f_2}{k}\right)\frac{1}{m} \text{ and } v = -\left(\frac{f_1f_2}{k}\right)m$$

Hence we see that all the forms remain the same as before, but the values of the focal length and of the positions of the principal foci are changed.

The new focal length is $\left(\frac{f_1f_2}{k}\right)$

The first principal focus moves in the positive direction a distance of $\frac{f_1^2}{k}$ and the second principal focus is $-\frac{f_2}{k}$ in the positive direction from the second principal focus of the second lens.

We may now add a third lens, but the forms obtained will obviously not be altered.

The focal length of any combination will obviously be

$$\frac{f_1 f_2 f_3 f_4, \text{ etc.}}{k_1 k_2 k_3, \text{ etc.}}$$

When k_n means the distance between the second principal focus of the n lenses and the first of the $\overline{n+1}^{\text{th}}$ lens.

A curious result of this which is at once apparent is that if any of the quantities k is equal to zero the whole focal length is infinite.

If the individual lenses have all negative focal lengths, but the values of k are all positive, the sign of the resulting focal length changes for each additional lens.

Thus it has been shown how, when we are in possession of the positions of the principal foci and focal length of two or more lenses, we can calculate the positions of the principal foci and the focal length of combinations of these lenses for particular positions of them.

It has also been shown how the positions of the principal foci of a single lens and its focal length depend upon the primary or structural quantities, viz. the two radii of curvature of the surfaces, the distance between these surfaces, and the refractive index of the material.

We have therefore a series of optical details dependent upon a series of structural details. In practical optics it is important that the optical properties of lenses should be known with great exactitude, and in another chapter I shall describe various modes of experimentally determining them. And assuming them known, it will be easy to find, from the formulæ already described, the structural details of the lens.

To this matter we shall in due time return, but in the meantime two special cases of lens combination will be considered, viz. the telescope and the microscope.

TELESCOPE.

If two lenses or lens systems are so situated that the second principal focus of one system is coincident with the first principal focus of the second system, the whole arrangement is called a telescope.

It derives its name from the fact that to a high degree of exactness it is that arrangement by which distant objects may be presented to the eye as subtending a larger angle than in reality the objects themselves do.

Let A B C be the principal foci of the lenses, B being, by definition, common to the two lenses, and let f_1 f_2 be the focal lengths.

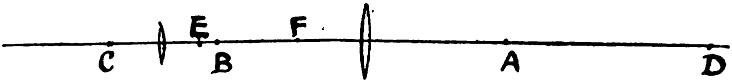

FIG. 23.

The distance represented in a former chapter by k as the distance between the second principal focus of the first lens, and the first principal focus of the second, is in this case equal to zero. Consequently the values $\frac{f_1 f_2}{k}$, $\frac{f_1^2}{k}$, and $\frac{f_2^2}{k}$ are all equal to infinity. This implies that the focal length of the arrangement is infinite, and that the positions of the principal foci are at an infinite distance from the system. In this case, therefore, we can draw no diagram, as in the general case. We may, however, infer that as the focal length is infinite, it will be necessary to move the object and image through infinite distances to obtain a *finite* change in the magnification, or in other words, that the magnification is constant. This we shall prove more strictly.

Let a small object be situated at D, and let the first lens produce an image of it at E. Then if m is the magnification of this image,

$$\text{B E} = -m f_1 \qquad \text{A D} = \frac{1}{m} f_1$$

Similarly if the second lens produces an image of E at F, and n is the magnification,

$$\text{C F} = -n f_2$$
$$\text{B E} = \frac{1}{n} f_2$$

Equating the two values of B E we have

$$-m f_1 = \frac{1}{n} f_2$$

$$\text{or } nm = -\frac{f_2}{f_1}$$

but nm must be the resulting total magnification (M).

$$\therefore \text{M} = -\frac{f_2}{f_1}$$

or the magnification of the image is independent of the position of the object, is constant, and equal to the relation of the second focal length to the first, with a negative sign.

Taking the other two equations we have

$$\frac{\text{AD}}{\text{CF}} = -\frac{f_1}{mnf_2} = -\frac{f_1}{f_2\text{M}} = \frac{f_1^2}{f_2^2} \therefore \frac{\text{CF}}{\text{AD}} = \frac{f_2^2}{f_1^2} = \text{M}^2$$

which equation gives us the relation of the distance of the image from C to the distance of the object from A.

Now when the arrangement is used to view distant objects A D is practically equal to C D, and the true way of describing what happens in a telescope is to say that the image is reduced in the ratio $\frac{f_2}{f_1}$, but is at the same time brought nearer to the eye in the square of that ratio, so that the angle subtended at the eye is greater than with unassisted sight in the ratio $\frac{f_1}{f_2}$. This value is often called the magnifying power of the telescope. It must not be confounded with the magnification of the image to which it is exactly inverse.

As $\frac{f_1}{f_2}$ represents the advantage of the instrument, telescopes are made by combining one system of large focal length with another of small, and turning that which has the larger towards the object, applying the eye at the end nearest the lens of shorter focal length.

Example.—Suppose the first or object glass has a focal length of 100 centimetres, and that the second or eye-piece has a focal length of 2 c.m., the magnifying power of the instrument is 50.

Suppose, also, that the object is 1 kilometre distant from the observer. The image formed will be at a distance of $(\frac{1}{50})^2$ of 1 kilometre or 40 centimetres only from the second principal focus of the eye-piece, and therefore not quite that distance from the eye itself. For many persons so near an aspect will not present a clear view, and an adjustment of the position of the eye-piece becomes necessary. A clear view of a distant object in a telescope is not, therefore, a guarantee that the instrument is in the *telescopic* condition of having two principal foci coincident. In fact two persons successively using the same instrument will make different adjustments. The true test of the *telescopic* condition is the *constancy of magnification* for differing positions of the object.

As the magnification is equal to $-\frac{f_2}{f_1}$ the image will be inverted if the two systems, object glass and eye-piece, are of the same character.

If, as in Galileo's telescope, f_1 and f_2 are of different signs the image is erect. What is called the day eye-piece of an ordinary telescope is a combination of glasses having a positive focal length, as a simple concave lens has. It is immaterial whether the lenses additional to the ordinary eye-piece be considered part of the object glass or of the eye-piece, or indeed where we choose to consider the tube divided in this respect. If we consider them as part of the object glass system, they will with the object glass form a system of positive focal length. Wherever a telescope in telescopic adjustment is divided, the focal length of one part bears to that of the other a constant ratio, and two principal foci, the second of the first system and the first of the second system, must coincide.

MICROSCOPE.

The arrangement of lenses for a microscope is one in which two systems are separated so as to have the value of K very large and positive. The two systems are generally convex.

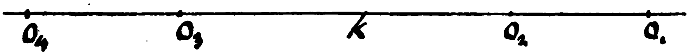

FIG. 24.

Let them have for focal length f_1 and f_2 each as a rule negative in value, and let o_1 o_2 o_3 o_4 be the principal foci, and the distance $o_2 o_3$ $(=k)$ be positive.

Then the focal length of the system is $\frac{f_1 f_2}{k}$, a positive quantity. Thus the system will be concave. The distance of the principal focus for the entering light will be situated to the right of o_1 by a distance equal to $\frac{f_1^2}{k}$, which will generally be a small quantity, k being large compared with f_1. Similarly the principal focus for the emerging light will be to the left of o_2 by the quantity $\frac{f_2^2}{k}$. So that the diagram for the whole arrangement will be one concave in character, of short focal length, and with the first principal focus very far to the right of the second; in this last particular widely differing from the diagram of a single thin concave lens.

The diagram will be of the sort indicated in accompanying figure.

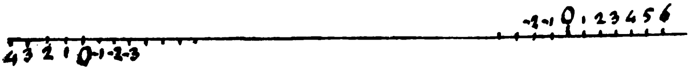

FIG. 25.

In this case if the object is placed to the right hand of the first principal focus the emergent light will be conveyed to a point to the left of the second principal focus, *i. e.* to the left of the eye-piece. The image would be real and erect, suitable for photographic purposes, but unfit for viewing by the eye.

For the latter purpose the image must be at a considerable distance to the right of the second principal focus; or the magnification must be negative, and of large numerical value, and the image virtual.

Hence the object must be placed at a short distance to the left of the first principal focus. But it must not be placed too far even on that side, for if it were to reach the first principal focus of the first lens the emergent light would be convergent to the second principal focus of the second lens, and therefore be unsuitable. The limits of motion, therefore, are less than $\frac{f_1^2}{k}$, but will otherwise depend upon the qualities of the eye.

Practically there is only one position of the object which will give a good view to the eye, and with high-power instruments, in which $\frac{f_1 f_2}{k}$ is very small as well as $\frac{f_1^2}{k}$, this fact necessitates very perfect slow motions in the adjustment to be made by an observer. One advantage of this is that when we see anything by means of such an instrument we may be sure that within excessively narrow limits the light encountering the instrument is divergent from a point at *a fixed distance* from the instrument.

A simple convex lens may be employed as a microscope, the advantage of which in this respect is greater the shorter the focal length.

In this case the points 1 and 1 of the two scales are never far apart, and the lens is also in the same region.

For distinct vision the image must be formed some distance to the right of the second principal focus. It will be therefore much magnified, erect and virtual. The object, therefore, must be placed minutely to the left of the first principal focus. As in the former case of the compound microscope, its position for clear vision of the image admits only of very small variation for one and the same eye.

CHAPTER VIII

COMBINATION OF A LENS AND A MIRROR

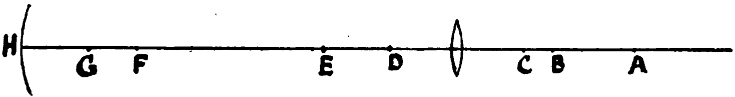

FIG. 26.

IN the diagram let

A be the position of an object in the common axis of a lens and mirror,

C and D be the first and second principal foci of the lens,

G be the principal focus of the mirror,

H be the mirror,

E be the position of the first image formed by the lens,

F be the position of the image of the last point produced by the mirror,

B be the image of F produced by the lens.

Also let f be the focal length of the lens,

r be the radius of the mirror,

k the distance ($=$ DG) between the second principal focus of the lens and the principal focus of the mirror,

$x =$ CA,

$y =$ CB,

$v =$ DE,

$w =$ GF,

M be the magnification of the final image at B relatively to the object at A.

The following equations hold good, viz. :—

$$xv = f^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\overline{k-v}.w = \frac{r^2}{4} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\overline{k-w}.y = f^2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the magnification of the final image is the product of the three successive magnifications of the images at E, F, and B, *i.e.*—

$$\text{M} = \left(-\frac{v}{f}\right)\left(-\frac{w}{\frac{r}{2}}\right)\left(-\frac{y}{f}\right)$$

$$= -\frac{2v.w.y}{f^2r} \quad . \quad . \quad . \quad (4)$$

Similarly $$\frac{1}{\text{M}} = -\frac{2.\overline{k-v}\;\overline{k-w}\;x}{f^2r} \quad . \quad (5)$$

If from the equations (1) (2) (3) (4) we eliminate v, w, x we obtain the equation

$$\text{M} = -\frac{\left(k^2-\frac{r^2}{4}\right)}{\frac{f^2r}{2}}\left\{y - \frac{f^2k}{k^2-\frac{r^2}{4}}\right\}$$

and from (1) (2) (3) (5)

$$\text{M} = -\frac{\left(k^2-\frac{r^2}{4}\right)}{\frac{f^2r}{2}}\left(x - \frac{f^2k}{k^2-\frac{r^2}{4}}\right)$$

Finally multiplying these values of M and $\frac{1}{\text{M}}$ together we have the connection between x and y, viz.—

$$\left(x - \frac{kf^2}{k^2-\frac{r^2}{4}}\right)\left(y - \frac{kf^2}{k^2-\frac{r^2}{4}}\right) = \left\{\frac{\frac{f^2r}{2}}{\left(k^2-\frac{r^2}{4}\right)}\right\}^2$$

The image formation therefore is that of a sphere whose focal length or semi-radius is—

$$\frac{f^2\frac{r}{2}}{\left(k^2-\frac{r^2}{4}\right)}$$

Its principal focus is distant from the first principal focus of the lens

$$\frac{kf^2}{k^2-\frac{r^2}{4}}$$

in the direction against that of the propagation of light before reflection.

Its centre is therefore distant from the same point in the same direction

$$\frac{kf^2}{k^2-\frac{r^2}{4}}+\frac{f^2\frac{r}{2}}{k^2-\frac{r^2}{4}}$$

$$\text{or } \frac{f^2}{k-\frac{r}{2}}$$

And its surface is distant from the same point in the same direction

$$\frac{kf^2}{k^2-\frac{r^2}{4}}-\frac{f^2\frac{r}{2}}{k^2-\frac{r^2}{4}}$$

$$\text{or } \frac{f^2}{k+\frac{r}{2}}$$

Now as the surface of the real mirror is situated at $\left(k+\frac{r}{2}\right)$ from the second principal focus of the lens, the surface of the virtual or resultant mirror being at $\dfrac{f^2}{k+\frac{r}{2}}$ from the first principal focus of the lens is conjugate to it. Similarly the centre of the mirror being at $k - \frac{r}{2}$ from the second principal focus of the lens, the centre of the virtual mirror being at $\dfrac{f^2}{k-\frac{r}{2}}$ from the first principal focus of the lens is conjugate to it.

Hence we have the simple rules governing the system composed of a lens and a mirror.

The virtual mirror's surface is conjugate with the real mirror's surface by the lens.

The virtual mirror's centre is conjugate with the real mirror's centre by the lens.

A number of interesting problems attach to these facts.

If the virtual mirror is plane, either the real mirror or its centre must be in the second principal focus of the lens.

If the mirror itself is so situated the virtual mirror will be at infinity, but its centre will not.

The position of the latter is distant from the first principal focus of the lens $\dfrac{f^2}{k-\frac{r}{2}}$, and k of this problem is $-\frac{r}{2}$, therefore the distance is $-\frac{f^2}{r}$ against the direction of light, *i.e.* $\frac{f^2}{r}$ towards the lens if the mirror is concave (r positive), otherwise if the mirror is convex (r negative). Under these circumstances, at this point if an object be placed it will coincide with an unmagnified but inverted image of itself.

In any case the image of an object will be inverted and unmagnified, and at this particular point they will coincide in position.

In the second case of the mirror's centre coinciding in position with the second principal focus of the lens, the centre of the virtual mirror will be at infinity, but the virtual surface will be (since now $k = \frac{r}{2}$) at a distance from the first principal focus of the lens $\frac{f^2}{r}$ away from the lens.

Under this condition the image will always be erect and unmagnified, and at this particular point will coincide in position with the object.

In the first and second cases the motion of object and image is always equal but opposite.

If one surface of a lens is silvered, and light is by this means reflected back through the lens after having entered it at the unsilvered side, we have a case of importance.

In this case the radius of the lens surface is the radius of the mirror, and the k of the problem is $v_0 - \frac{r}{2}$.

Hence the centre of the virtual mirror is distant from the first principal focus of the lens $\frac{f^2}{v_0 - r}$ in a direction away from the lens.

The focal length of the virtual mirror is

$$\frac{f^2\frac{r}{2}}{\left(v_0 - \frac{r}{2}\right)^2 - \frac{r^2}{4}}$$

or

$$\frac{f^2 r}{2v_0(v_0 - r)}$$

An object placed at the centre of the virtual mirror will coincide in position with its image, and this will form in practical optics an easy criterion of an object being so situated.

CHAPTER IX

STRUCTURAL DETAILS OF LENSES IN TERMS OF OPTICAL DETAILS

In Chapter VI. we have investigated the ways in which what may be called the optical properties of a lens, u_0, v_0, and f_1, depend upon the structural elements of a lens, viz.—$r_1\ r_2\ d$ and μ. By means of the formulæ obtained we can calculate the former when the latter are given.

It is, however, quite possible to obtain formulæ which give $r_1\ r_2$ and μ in terms of $u_0\ v_0\ f$ and d, so that if these latter are determined experimentally, $r_1\ r_2$ and μ may be separately investigated.

Such formulæ would be of use in testing a lens to see that the specification had been truly adhered to. They would also be a means of investigating the index of refraction of a material in the shape of a lens.

They of course result from the proper algebraical treatment of the equations already established. We therefore give the results without further preamble.

$$\mu = \frac{-fd}{f^2 + u_0 v_0}$$

$$r_1 = \frac{f^2 + u_0 v_0 + fd}{v_0 - f}$$

$$r_2 = \frac{f^2 + u_0 v_0 + fd}{u_0 + f}$$

We shall now show how the quantities involved on the right-hand sides of these equations may be obtained experimentally for a simple convex lens.

The thickness of the lens *d* may be found directly by measurement with a callipers furnished with the properly-shaped jaws and with a vernier. u_0 and v_0 may easily be measured, if the former is positive and the latter negative, as is generally the case in single lenses of convex character, *i. e.* of negative focal length, in the following way. Place the lens upon one of its faces on a plane horizontal mirror. Cut a sharp point out of a piece of stiff white paper, and place it vertically over the centre of the lens, holding it by means of a universal holder or other apparatus allowing up-and-down motion. When the paper is at the principal focus, the light from it after passing through the lens will be in parallel beams. Hence it will be reflected back from the mirror in parallel beams, and therefore will be brought again to a focus in the principal focus, *i. e.* where the object is. This may be tested, either by the unassisted eye observing whether there is any distance between object and image, or by using a microscope or magnifying lens of short focal length to view the object. By moving this instrument backwards and forwards it will be found not difficult to say if the image and object are not coincident. One will come into focus and go out of focus before the other if they are not at the same distance from the lens. When the adjustment is satisfactory, it is only necessary to measure from the paper point to the nearest face of the lens. This measurement will be u_0 or $-v_0$ according as the first or second surface is upwards. Thus the positions of the principal foci are determined. To find *f*:—

From a small telescope take out all the lenses except the object glass, and place it vertically upon a horizontal plane mirror, object glass downwards. Lay a plate of glass across the open end of the tube, and place a small definitely-cut piece of white paper upon the plate. The smaller the piece of paper the better; three or four square millimetres in the shape of a triangle are sufficient. Then adjust the tube of the telescope until the image of the paper coincides with the paper itself, testing with a magnifying glass. This adjustment being complete, measure the length of the tube: suppose it to be l_1.

Now place the convex lens, the position of whose principal foci we already know, upon the plate of glass at the end of the tube, but put the small piece of paper upon the top of the lens. Then adjust the tube again until the paper again coincides with its image. Measure again the length of the tube: suppose it to be l_2.

Then it can be proved very easily that

$$f^2 = -u_0 \{ v_0 - \overline{l_1 - l_2} \}$$

The first observation clearly gives the position of the principal focus of the object glass of the telescope and the piece of plate glass considered as one system. This will not be altered by merely shifting the plate glass nearer to the object glass, *i. e.* downwards in the case supposed. The shortening of the tube (l_2 will in general be less than l_1) therefore merely has the effect of placing the principal focus of the object glass and plate at a distance within the lens equal to the amount of shortening. Therefore the distance (k of former formulæ) between the second principal focus of the lens whose focal length we are seeking, and the first of the combination formed of the object glass and plate, is $v_0 - \overline{l_1 - l_2}$

or $$k = v_0 - \overline{l_1 - l_2}$$

But the first principal focus of the lens is u_0 from the lens, and in the complete combination it is on the upper surface. This difference is $\frac{f^2}{k}$ measured contrary to the direction of light. Numerically it is equal to u_0, as the principal focus of the combination is on the surface of the lens.

Therefore $$\frac{f^2}{k} = -u_0$$

Hence $$\begin{aligned} f^2 &= -u_0 k \\ &= -u_0 \{ v_0 - \overline{l_1 - l_2} \} \end{aligned}$$

as stated above.

In this formulæ v_0 has a negative value in general and $\overline{l_1 - l_2}$ is positive, so that the whole expression for f^2 is positive, as it should be.

The reason for introducing the plate of glass will appear from the following statement.

If the lens which is being measured has one flat side, we may proceed without the glass plate, stretching two silk fibres across the open telescope tube instead of using the paper. The tube is then adjusted till the cross fibres coincide with their own image, when l_1 is measured. The lens may then be laid with its flat side upon the tube, the piece of paper being applied to the convex side, and the tube adjusted and measured for l_2. This would clearly give an accurate result, and is in practice very convenient to manage. But if the under side of the lens is convex and this were laid upon the open tube directly, it would bulge into the tube by an uncertain amount, and we should have incorrect quantities to deal with. The placing of the plate will answer so long as the under side of the lens is not concave. Should it be concave there is at least one side which is not so, and that can be employed upon the plate. Having determined f by this method, the sign to be applied to the number obtained must be settled by other considerations.

To make the practice of this method clearer, I give an example.

An achromatic lens having one flat side had its principal foci determined to be 15·4 c.m. from the convex side and 14·4 c.m. from the flat side.

The shortening of the tube employed was ·7 c.m., and the flat side of the lens was downwards.

Here therefore $u_0 = 15{\cdot}4$

$$v_0 = -14{\cdot}4$$

$$l_1 - l_2 = {\cdot}7$$

$$\therefore f^2 = -15{\cdot}4\{-14{\cdot}4 - {\cdot}7\}$$

$$= 15{\cdot}4 \times 15{\cdot}1$$

$$\therefore f = -15{\cdot}25 \text{ c.m.}$$

the negative sign being applied because the character of the lens in this respect was obvious. The thickness of this lens was 1·2 c.m.

Therefore the distance between the principal foci was

$$15{\cdot}4 + 1{\cdot}2 + 1{\cdot}44 = 31 \text{ c.m.}$$

Adding twice the focal length 30·5

we have 61·5 c.m. as the distance between the conjugate points for an inverted unmagnified image. Approximate methods direct that this must be divided by 4 to obtain *f*. We should thus obtain the number 15·375 instead of 15·25, an error of about ·82 per cent. in excess.

The apparatus required for the practical operation described above is of the simplest character. If an optical bank is available, methods depending on magnification observations can be employed with great advantage.

An optical bank is an apparatus constructed of wood or metal upon which different pieces of apparatus such as lenses, telescopes, microscopes or magnifying glasses, screens for receiving images, and transparent scales, may be moved backwards and forwards relatively to each other, but always in one straight line. Such motions may, by means of a scale running parallel with the line of motions, be easily measured.

To employ magnification methods, we must have the means of readily observing the relative size of an image and object. This is obtained in having two divided scales, equal in every respect, upon transparent glass. They may be photographic positives taken off the same negative.

To obtain the optical details $(u_0 v_0 f)$ of a convex lens, or of a convex combination such as those employed in magic-lanterns, in cameras, in eye-pieces of microscopes, or in the astronomical eye-pieces of telescopes, let the lens or combination be placed in an appropriate holder somewhere near the middle of the optical bank, and one of the ruled scales in its carrier near one end of the bank, taking care that the glass upon which the scale is ruled is *not* between the scale and the lens. Then the other scale, turned likewise with the ruling next the lens, must be so placed on the other side of the lens that it may receive in good definition the image of the distant scale. If necessary light must be sent through the whole arrangement, by reflecting it from the sky, or from a lamp.

The adjustment of this second scale must be made very carefully. A short-focus convex lens used in the way described on p. 66 is an admirable means of testing this adjustment.

When this adjustment is complete we have two scales upon the same plane—one the actual scale upon the near glass, the other the image of the distant scale.

Suppose that p divisions of the former are equal in length to q of the latter, then the actual magnification of the image is $\frac{p}{q}$. This relation therefore is the subject of the immediate observation.

As the two screens are now in conjugate points, we may, as a useful check upon the value of the magnification, send the light the other way through the apparatus and take the magnification at the first scale, which should be the inverse of that first determined.

We have got by this means, if we measure the distances of the scales from the lens, the positions of two points upon the diagram, one upon that portion which refers to light before entering the lens, and the other for the emergent light, to which we can attach the appropriate numbers (with negative signs).

We can now push the first scale nearer to the lens, and repeat observations, obtaining another pair of points. This will complete the diagram, the focal length f being furnished from the motion of either screen and the change in the magnification number, for this latter is the quotient obtained by dividing the motion by the focal length.

However, the process can be continued at intervals until the second scale reaches the limits of the optical bank. Very great accuracy may be obtained by combining the various results in the ways which experimentalists will understand, and focal lengths as high as twenty or thirty centimetres may be determined to one-tenth of a millimetre.

Another mode of procedure can be adopted. After the first step described above is completed, the distances measured, and the magnification m determined, instead of moving the scales, one may push the lens system itself towards the first scale until clear definition is again obtained upon the scales. The magnification then is equal to $\frac{1}{m}$, and therefore has changed by the amount $\left(m - \frac{1}{m}\right)$. The length through which the lens has moved must then be noted (l).

And we have the equation,

$$f = \frac{l}{m - \frac{1}{m}} = l\,\frac{m}{m^2 - 1}$$

It will be noted that by this formula the *numerical* value of f will be the same whether we employ the magnification observed or its inverse for m. Thus suppose the observed m is -3, and $l = 4$. Then if we put $m = -3$ we obtain

$$f = -\frac{3}{2}$$

and if we put $m = -\frac{1}{3}$ we have

$$f = \frac{3}{2}$$

Where the proper sign is, on other grounds, obvious, this fact saves a little trouble in practice.

Concave lenses do not lend themselves to this operation directly, but a modification of it, to be described after some auxiliary methods have been touched upon, will enable us to determine any number of points in the diagram of such a lens.

For many measurements a most convenient instrument is a microscope with a ruled scale, situated between the object glass and the eye-piece. We can determine the values of the dimension corresponding to one division of the scale by employing the instrument upon a known dimension, such as a scale of tenths of a millimetre. After such a determination we can always employ the microscope upon any small object, and determine the dimensions of it by reading off how many divisions of the scale inside the microscope its image occupies. Such an instrument is called a measuring or micrometer microscope. Care must be taken not to change the microscope in any way, either by employing different object glasses, or by lengthening the tube so as to change the distance apart of the lenses.

In some applications of this instrument it will be found that it is not necessary to know the value of the divisions of the microscope scale.

To find the relation of the focal lengths of the object glass and eye-piece of a telescope.

If both these lenses are convex, they may be found separately by methods already described.

In any case there are other methods applicable. When the telescope is in what I have called the telescopic condition, this relation is also the ratio which an object bears to its image in linear dimensions, for light proceeding in the usual way through the instrument. With a measuring microscope, therefore, we can then obtain the dimensions of the image of an already measured object, and so obtain the relation required. But for this we must be sure that the telescopic condition obtains, and adjust the telescope if it does not do so.

A good test depends upon the fact that in this condition the relation in size of object to image is constant wherever the object is placed. A number of observations of the image may therefore be made, the object occupying *different* positions. If the number of divisions of the microscope occupied by the image is always the same, the condition required has been reached.

Approximately the condition may be reached by viewing through the telescope any heavenly body, and adjusting to distinct vision. With short-sighted persons this will not give a very close approximation.

In what follows a second trustworthy method is indicated, depending upon the travel of the principal focus of a lens resulting from the addition of another lens to the system.

Suppose the position of the second principal focus and the focal length of the eye-piece to be known. They can in convex eye-pieces be determined as already shown.

Now, suppose that it is desired to form an image of a heavenly body (the sun or moon) on a plane at a known distance from the telescope, as in celestial photography. Let l be the distance from the second principal focus of the eye-piece and this plane. This will, therefore, be the travel of the principal focus resulting from the addition to the system of the object glass of the telescope. If, therefore, k is the distance between the second principal focus of the object glass and the first of the eye-piece, and f the focal length of the eye-piece,

$$l = \frac{f^2}{k} \text{ or } k = \frac{f^2}{l}$$

If, therefore, l is known k becomes known, and *vice versâ*, if k is known l becomes known.

Now had the total length of the telescope in the telescopic condition been known, and our object been to produce an image of a heavenly body on a screen at distance l from the second principal focus of the eye-piece, we should lengthen the tubes of the telescope by $\frac{f_2}{l}$.

Conversely, if we wish to put the telescope into the telescopic condition, obtain a clear image of a heavenly body on a screen whose distance from the second principal focus of the eye-piece is l and known, and then rack in the eye-piece by the distance $\frac{f^2}{l}$. It will then be in the condition required.

The change of length will not in general be found to be very large. I give an example from an actual case.

The focal length of the Huyghen's eye-piece of a telescope was $-2{\cdot}446$ c.m., and the second principal focus was situated outside the system at a distance ·75 c.m. from the outside glass surface of the eye-piece.

The plane of the focussing plate of the camera attached to the instrument was found to be 13·46 c.m. from the outside glass surface of the eye-piece.

Therefore $l = 13{\cdot}46 - {\cdot}75 = 12{\cdot}71$ c.m.

and $k = \frac{f^2}{l} = \frac{\overline{2{\cdot}446}^2}{12{\cdot}71} = {\cdot}471$ c.m.

A good image of the moon was obtained upon the camera plate; the camera was then unscrewed, and the length of the telescope over all was measured and found to be 109·54 c.m. In the telescopic condition, therefore, the telescope measures over all

$$109{\cdot}54 - {\cdot}47 \text{ or } 110{\cdot}07 \text{ c.m.}$$

In this condition it is found to give an excellent definition of heavenly bodies to far-sighted eyes, but not otherwise.

To find the position of the principal foci and the focal length of a concave lens.

Place a telescope in telescopic adjustment on the optical bank, and near the eye-piece end of it place a measuring microscope.

Beyond the object glass end place a well-defined object, such as a piece of nickel sheet cut with sharp scissors so as to present two clean points, as in the sketch.

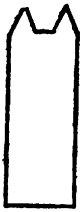

FIG. 27.

By moving the measuring microscope obtain a clear image of this, and count how many divisions of the scale it occupies. Let this number be p. Attach the concave lens by a little wax or otherwise to the object glass end of the telescope. Obtain a clear image again by moving the microscope, and read the divisions occupied. Let this be q. Then it is clear that the image of the object formed by the concave lens bears to the object the relation $\frac{q}{p}$, or, in other words, the point which the object occupies relatively to the concave lens must bear the diagram number $\frac{p}{q}$. Let this position, therefore, be measured and recorded. Now shift the position of the object by any convenient distance l, and repeat the observation. Suppose now r divisions of the microscope scale are occupied by the image. The object now is in the point whose diagram number is $\frac{p}{r}$, and so any number of points may be obtained for the side of the lens turned to the object. Moreover $\frac{p}{r}-\frac{p}{q}$ is the change in the size of the object relatively to its image due to a change in position l. Hence if f is the focal length of the concave lens,

$$f=\frac{l}{\frac{p}{r}-\frac{p}{q}}=\frac{l}{p}\cdot\frac{rq}{q-r}$$

To obtain points on the other side of the lens, this must be reversed upon the telescope, and similar observations made.

The thickness of a simple lens can of course be best measured by actually gauging it with callipers or micrometer screws.

We have seen how in general the positions of the principal foci and the focal length can be found by optical experiments. With the further knowledge of the thickness of the lens, in the case of simple lenses, all the remaining structural details can be found, viz. the two radii of curvature and the index of refraction, by the formulæ already given, viz.—

$$r_1 = \frac{f^2 + u_0 v_0 + f d}{v_0 - f}$$

$$r_2 = \frac{f^2 + u_0 v_0 + f d}{u_0 + f}$$

$$\mu = -\frac{f d}{f^2 + u_0 v_0}$$

It will, however, be noticed in these forms that with ordinary thin lenses there would be great liability of error in attempting to use them for determing r_1 r_2 and μ. This is because the denominator is sure to be a very small quantity.

With a convex lens f and v_0 are negative and u_0 positive, while numerically they are nearly equal.

With a concave lens f and v_0 are positive and u_0 negative.

A better mode of determining the radii of the lens faces when convex is as follows :—

Having determined u_0 v_0 and f in the ways already indicated, place the lens with its second surface on a pool of mercury, and find the centre of the virtual mirror experimentally.

It has been shown that this will lie in a position removed from the first principal focus of the lens a distance of $\frac{f^2}{v_0 - r_2}$ in a direction contrary to that of the light before reflection. If, therefore, c is this travel of the principal focus due to the mirror,

$$c = \frac{f^2}{v_0 - r_2} \quad \text{or} \quad r_2 = v_0 - \frac{f^2}{c} \quad .$$

Similarly r may in its turn be determined. These formulæ are not open to the objection mentioned in the case of those involving merely $u_0\, v_0\, f$ and d, and the method is exceedingly easy in practice. It does not assist, however, in determining the index of refraction more exactly.

If the lens is not very thin the formula

$$\mu = -\frac{f d}{f^2 + u_0 v_0}$$

is more trustworthy.

With a sphere, for instance, we should have from this:

$$\mu = 1 - \frac{d}{d + 4f}$$

or since in this case $u_0 = -\left(f + \frac{d}{2}\right)$ (the two surfaces being concentric),

$$\mu = 1 + \frac{d}{4\,u_0 + d}$$

either of which forms are good for practical employment.

To find the index of refraction of a liquid by means of a convex lens and plane mirror.

When a convex lens is placed upon a plane mirror, we may find, as already explained, its principal focus by the fact that at that point an object will coincide with its image.

Suppose, therefore, that this has been done for both the principal foci, and that further the focal length of this lens has been determined. There is no need for the lens to be simple. It may be an achromatic lens or a compound photographic objective.

Now let a few drops of a liquid whose index of refraction is sought be placed between the lens and the plane mirror, and let the principal focus of this combination of liquid and lens be found in the same way, and suppose that c is the travel of the principal focus due to the insertion of the liquid lens, upwards or away from the system.

If r is the radius of the surface common to the lens and the liquid, and if μ is the index of refraction of the liquid, the liquid lens will have for focal length $\frac{r}{\mu - 1}$, for its thickness vanishes, and its first principal focus will be situated at $\frac{r}{\mu - 1}$ below the plane of the mirror.

The distance (k) between the second principal focus of the solid lens and the first of the liquid lens is therefore

$$v_0 + \frac{r}{\mu - 1}$$

but

$$\frac{f^2}{k} = c$$

$$\therefore \frac{f^2}{v_0 + \frac{r}{\mu - 1}} = c$$

whence

$$\overline{\mu - 1} = \frac{r}{\frac{f^2}{c} - v_0}$$

We may now substitute for r its value—

$$\frac{f^2 + u_0 v_0 + fd}{u_0 + f}$$

in which case we should have an expression for μ involving only the optical elements u_0 v_0 f of the lens, its thickness d easily obtained, and the length c; but we have seen that unless the lens is thick this expression for r is not adapted for making an accurate determination of that quantity. It is better to determine r by means of a pool of mercury.

The formula giving μ may be written

$$\frac{1}{\mu - 1} = \frac{f^2}{rc} - \frac{v_0}{r}$$
$$= \frac{\text{A}}{c} + \text{B}$$

when c varies with μ, but A and B are constants involving properties of the lens alone.

A and B may be determined experimentally by two observations of c made when μ has values known otherwise.

With this method care must be taken that the combination of solid and liquid lens is convex. The liquid lens must not be stronger (numerically) than the solid one. If, therefore, highly refracting liquids are to be investigated, the glass of the lens should be dense, or its upper side convex, as well as the other, which must be so.

From the form it will be seen that as $\overline{\mu - 1}$ increases, so must c, and the mathematical limit to the value of μ which can be measured with any convex lens occurs when c becomes infinite, in which case

$$\overline{\mu - 1} = \frac{1}{\text{B}} = -\frac{r}{v_0}$$
$$\text{or } \mu = 1 - \frac{r}{v_0}$$

CHAPTER X

MICROSCOPE METHODS APPLIED TO LENSES

I SHALL now describe some useful applications of the micrometer microscope to the determination of focal lengths, these being moderately small, or of the order of a few centimetres. This method is especially applicable to concave lenses, such as occur in the eye-pieces of opera-glasses, but will also serve for convex lenses, or combinations such as occur in the eye-pieces of astronomical telescopes and microscopes.

Place any small object upon the stage of the microscope. A millimetre divided into tenths, with an extra tenth divided into hundredths, is an admirable object. Any number of the divisions may be selected as the object; the higher the power of the microscope the smaller would be the number selected, so as to make sure of the selected numbers not being too many for the scale or field of view within the microscope.

(1) Observe the number of divisions of the micrometer subtended by the image of the object, when viewed directly, and let this number be a.

(2) Place the lens to be tested over the object, and observe again, the number subtended being b.

(3) Insert a tube of known length (l) determined by callipers between the lens and the object, focus again, and observe the divisions (c) of the scale.

Then
$$f = \frac{l\,b\,c}{a\,(b-c)}.$$

For the size of the object relative to that of the image is in the first case $\frac{a}{b}$ and in the second case $\frac{a}{c}$, whence $a\left\{\frac{1}{c}-\frac{1}{b}\right\}$ is the change in this relation, and, on the principles explained, must be equal to $\frac{l}{f}$; whence the formula above given.

Suppose the distance by which the microscope tube moves relatively to the lens in the above operation is also noted. If the instrument carries a vernier moving over a scale, the reading of the vernier in the two positions can be made, the difference being the actual motion of the microscope tube. But the lens has been moved through the length l, so that the actual approach of the tube to the lens will be the difference of the vernier readings increased by this length l. Let this length be L.

Then clearly the relative size of the image to the object changes from $\frac{b}{a}$ to $\frac{c}{a}$.

Hence also $\frac{1}{a}(b-c)=\frac{\mathrm{L}}{f}$

or $f=\frac{\mathrm{L}\,a}{b-c}$.

Thus a check observation is furnished by noting the vernier readings on the microscope. Great confidence may be felt in the result when the two values of f so obtained agree.

If we multiply these equations together we have

$$f=\frac{\sqrt{\mathrm{L}\,l\,b\,c}}{(b-c)}$$

where (a) has disappeared. This measurement, therefore, is not necessary when the vernier readings have been taken so that L may be employed.

An important warning should be uttered at this point.

In observing *b* and *c*, note if the aspects of the object are identical or reversed relatively. If the divided millimetre is employed, note, for instance, if the extra tenth division is towards the same side, say to the right hand, or not.

If the image suffers reversal, as indicated by a reversed aspect, then the object on the stage has in its virtual motion from the lens passed through the principal focus. In such a case *c* and *b* must be considered to have different signs, and using the *numerical* values of *b* and *c* we must employ a positive sign between them. This is often the case with convex eye-pieces, but is not likely to occur with concave eye-pieces, except in the cases of day eye-pieces to telescopes.

To illustrate this matter, as well as to indicate another of the artifices which will always occur to those who have well grasped the theory of these magnification methods, I append a case, taken from my laboratory book, of a telescope eye-piece.

The object (which was one particular 6-tenths of a millimetre) appeared on direct view to occupy 19·4 divisions of the micrometer scale. Hence $a = 19{\cdot}4$.

The eye-piece was then placed over the object, and its image then occupied 43·8 divisions. The vernier reading of the microscope was 0·95 c.m.

A tube of 7·18 c.m. was now inserted between the eye-piece and the object, the image, which was reversed, occupying 5·9 divisions. To obtain a view of it the whole microscope tube required to be pulled back in its holder. A pin of known dimensions 10·005 c.m. being inserted between the brass collars, which would otherwise have been in contact, effected this easily. Upon adjusting the view the vernier read 3·06 c.m.

Here obviously the actual motion of the microscope tube was $3{\cdot}06 - 0{\cdot}95 + 10{\cdot}005 = 12{\cdot}115$.

From this to obtain the motion of the tube relatively to the lens the length of the tube under the lens 7·18 must be deducted. The remainder 4·935 is the L of the problem.

Here therefore we have

$a = 19{\cdot}4$
$b = 43{\cdot}8$
$c = 5{\cdot}9$ but reversed.
$l = 7{\cdot}18$
$\text{L} = 4{\cdot}935$
$\overline{b - c} = 49{\cdot}7$ remembering the reversal.

From the formula

$$f = \frac{\text{L}\,a}{b - c} \text{ we obtain } f = -\ 1{\cdot}926 \text{ c.m.}$$

$$\text{From } f = \frac{l\,b\,c}{a\,(b - c)} \quad ,, \quad = -\ 1{\cdot}924 \ ,,$$

$$\text{and from } f = \frac{\sqrt{\text{L}\,l\,b\,c}}{(b - c)} \quad ,, \quad = -\ 1{\cdot}925 \ ,,$$

Measurement of Microscope Objectives.

The objective of a microscope may be treated, like any other combination, in the way already described of finding short focal lengths. Under some circumstances, however, the objective may be screwed into its place in the micrometer microscope tube. In this case two things are necessary. The first is that the actual value between the divisions of the micrometer scale shall be known. The second is that nothing, unless it be a plate, shall intervene in the microscope tube between the objective and the scale. In many good eye-pieces a lens does so intervene, but in that case it may be removed for the time being. The process is the following :—

Observe a portion of a divided millimetre, with the draw tube of the microscope pushed home, so that the instrument is at its shortest. Let a be the real value of the object, b the real value of a single division of the micrometer's scale, and m be the number of the divisions subtended by the image. Then $\frac{m\,b}{a}$ is the size of the image relatively to the object.

Now pull the draw tube out a known distance. A distance-piece about 10 c.m. long, such as was mentioned in the operation described on p. 89, is convenient, but it must now be inverted, so as actually to lengthen the microscope tube. Let this elongation be l. Bring the image to focus again, and let n be the observed number of divisions subtended. Now $\frac{n\ b}{a}$ is the magnification, and therefore $\frac{\overline{n-m}\ b}{a}$ is the change in the magnification due to a motion l. Hence $f=\frac{l\ a}{\overline{n-m},\ b}$.

The microscope can be employed to find the focal length of a combination of considerable focal length, such as a photographic lens, when the position of the principal focus has been determined, say by the plane mirror method.

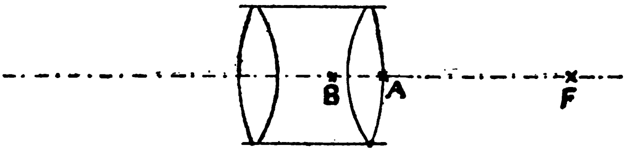

FIG. 28.

Suppose F the principal focus, and B distant from F by the focal length. B will be the point 1 of the diagram. Let A be the point where the axis cuts the surface of the lens system. Now F A is supposed to be known. Call it u_0.

Place a small object, such as a minute piece of tinfoil, upon the surface at A. A little breath will cause it to adhere.

Place the system upon the stage of the microscope with the small object turned towards the microscope; note the divisions (a) of the scale subtended.

Turn the system over and measure what will now be the image of the small object formed by the combination. Let the divisions now subtended be (b).

Then clearly the size of the object relatively to the image is $\frac{a}{b}$.

But the magnification at A is also $\frac{\text{FA}}{f}$ or $\frac{u_0}{f}$.

Hence $f = u_0 \frac{b}{a}$.

If b is greater than a, the point B would be inside the system, and *vice versâ.* In a properly constructed Huyghen's eye-piece, in which the larger lens is convex-plane, the point A is the point -1 of the diagram, and under such a test, of remarkably easy application, a small object placed in the middle of the convex surface of the larger lens should appear of the same size when viewed either way.

For photographic lenses this method is most strongly to be recommended.

An easy and ready way, being one which involves only a single observation, of finding the focal length of a lens is based upon the following proposition.

If a small object is placed in the principal focus of a lens of focal length f_1, and the emergent rays be received upon a second lens of focal length f_2, the magnification of the final image will be equal to $-\frac{f_2}{f_1}$, whatever may be the distance separating the lenses.

For let k as usual be the distance between the second principal focus of the first lens and the first principal focus of the second lens. Then the focal length of the combination is $\frac{f_1 f_2}{k}$.

But the principal focus of the combination will be at the distance $\frac{f_1^2}{k}$ from the position where the object is. Hence the magnification point of the spot where the object is situated is $-\frac{\frac{f_1^2}{k}}{\frac{f_1 f_2}{k}}$ or $-\frac{f_1}{f_2}$.

Therefore the magnification of the final image is $-\frac{f_2}{f_1}$, which establishes the proposition, being independent of k. This method may readily be applied to long focal length lenses, f_1 being the focal length of the lens of a collimator with an easily measured fixed object at its principal focus. A bank to carry the collimator, the lens to be measured, and a transparent ruled scale would be required.

But with a micrometer microscope the method is particularly adapted to short focal lengths, such as those of eyepieces and object glasses of microscopes. A collimator of about 3 c.m. focal length, and with a small hole or two fine parallel lines in its principal focus as an object, may be mounted in the hole of the stage of a microscope. Any lens to be experimented upon is then placed over this arrangement and the final image measured.

Then if b be the division of the scale which this image occupies, a the division occupied when the object is regarded directly, and f_1 the focal length of the lens of the collimator, the focal length f_2 of the lens under examination will be given by the equation—

$$f_2 = -\frac{f_1}{a} b$$

The quantity $\left(-\frac{f_1}{a}\right)$ is a pure function of the collimator, and when this value has been once established, the focal lengths of large numbers of lenses may be determined with great rapidity.

Short focal length concave lenses, such as are used in the eye-pieces of binocular telescopes, may be determined in this way if the object glass of the microscope is not too powerful; that is to say, the image being in the case of concave glasses virtual, the microscope must be able to reach it.

CHAPTER XI

FORMS OF LENSES FOR MINIMUM DEVIATION OF RAYS

In dealing hitherto with questions of image formation we have considered the pencils of light to be very small and indefinitely near the general axis of the system. In practice larger pencils are employed in order to obtain sufficiency of light. Rays which deviate from the axis will not come exactly to one focus, and the question arises how far this can be corrected by modifying the two radii of curvature. For while the focal length and thickness of a lens may remain fixed, there will still be freedom to dispose of these quantities, subject to the formulæ which connect them.

When a ray passes with minimum deviation through a prism, the angles of incidence and emergence are equal, and the total deviation is shared equally by the two surfaces. Under this condition any small change in the angle of incidence results in the same small change in that of emergence. If we write down the usual equation to a lens, multiplying each term by h, the distance from the central point of the lens to the point where any ray passes through it,

$$\frac{h}{u} - \frac{h}{v} = -\frac{h}{f}$$

$\frac{h}{u}$ and $\frac{h}{v}$ are approximately the angles which h subtends at the object and image respectively, and the equation states that their difference must be kept constant, which will be the case, the lens being supposed thin, if the sum of the angles of incidence and emergence remains the same. Hence in any particular case it is desirable that the ray should traverse the lens with minimum deviation. Of course this cannot be true for all positions of the object, but the condition can be secured for all rays emanating from any given point upon the axis. And in a large number of cases, as in the lenses of a telescope, for instance, the position of the images is always invariable.

In what immediately follows this point will be considered.

Let two circles whose radii are $r_1 r_2$ respectively be described with centres at A and B.

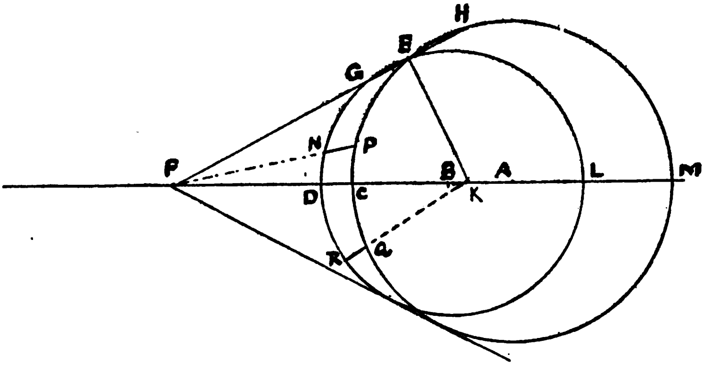

Fig. 29.

Let one of their points of intersection be E. Let F be the point of similarity for the two circles found by making a common tangent cut the line A B produced. Join F E, and let that straight line cut the circles in G, E, and H. Now it is clear that the angles which the straight line F H makes with the arcs G E, E H are all equal, and as C E is only the continuation of the arc E H, and E L of G E, it is evident that F H bisects the angles made by the pairs of arcs

G E, C E
E H, E L

Draw EK perpendicular to FH, and cutting AB in K. Then also EK will bisect the angles made by the pairs of arcs

CE, EL
GE, EH

The point K will be a point of negative similarity for the two circles, which merely means the following:—

If from K a point moves in any one given direction to reach either circle, then it must move in exactly the opposite direction to reach the corresponding point of the other circle, and the two distances traversed will be in the ratio of the radii of the circles.

The two points F and K are the extremities of the hypothenuse of a right-angled triangle, the right angle being at E, and EF, FK are the pair of bisectors of the angles formed by the arcs at E.

Let FNP be any straight line through F, cutting the circles in NP.

Then clearly NP cuts the arcs at equal angles, and if KQR be a straight line through K, RQ cuts the arcs it meets also at equal angles. The difference in the two cases is this, that the normals at the points of intersection drawn between the two arcs will be in one case on the same side of the intercepted portion of the straight line, and in the other case on opposite sides of the same.

If, therefore, the space between the arcs considered be the section of a refracting medium, the intercepts will represent portions of rays which suffer in one case minimum deviation, in the other case no deviation, the latter of course corresponding to the case of the normals being upon opposite sides of the intercept, and *vice versâ*.

Now we may have the refracting material in any one of the four compartments

(1) DEC
(2) LEM
(3) CEL
(4) DEM

These will constitute four different lenses having face curvatures of the same value.

In (1) (2) the point for no deviation is F, and that for minimum deviation is K.

In (3) and (4) the point for no deviation is K, and that for minimum deviation is F.

The point for no deviation has been called the centre of a lens, that for minimum deviation has not, so far as the author is aware, received any particular name. It may be called the minimum deviation point.

Both these points are purely geometrical matters, and have no dependence whatever upon the index of refraction. Their position can be found in terms of the two radii of the surfaces and the thickness of the lens ($r_1\, r_2\, d$).

Thus $$\frac{\mathrm{FC}}{\mathrm{FD}} = \frac{r_1}{r_2} \text{ and } \mathrm{FC} - \mathrm{FD} = d$$

Hence $$\mathrm{FC} = \left(\frac{r_1}{r_1 - r_2}\right) d, \qquad \mathrm{FD} = \left(\frac{r_2}{r_1 - r_2}\right) d$$

Similarly

$$\frac{\mathrm{KC}}{\mathrm{KL}} = \frac{r_1}{r_2} \text{ and } \mathrm{KC} + \mathrm{KL} + d = 2r_2$$

Whence

$$\mathrm{KC} = \frac{r_1}{r_1 + r_2}\left(2\,r_2 - d\right)$$

and

$$\mathrm{KD} = \frac{r_2}{r_1 + r_2}\left(2\,r_1 + d\right)$$

It remains to determine in what position upon the axis of a lens, say DEC, a small object must be placed in order that the rays refracted at the first surface shall have paths passing through (1) the point K, (2) the point F.

If u is the distance of the object from the surface CE, and v is the distance of the image, the two quantities are connected by the equation (p. 27), for refraction at one curved surface: $\frac{1}{u} - \frac{\mu}{v} = -\frac{\mu - 1}{r}$.

To apply this equation to the point K we must write r_1 for r, and $\frac{r_1}{r_1+r_2}\left(2r_2-d\right)$ for v.

Then
$$\frac{1}{u}=\mu\frac{\overline{r_1+r_2}}{r_1}\,\frac{1}{2r_2-d}-\frac{\overline{\mu-1}}{r_1}$$
$$=\frac{1}{r_1}\left\{\frac{\mu(r_1-r_2)+2r_2+\overline{\mu-1}\,d}{2r_2-d}\right\}$$

an equation by means of which we can determine the value of the relation of r_1 to r_2 in special cases. Suppose, for instance, that u is infinite, and d negligible,

Then
$$\mu\,(r_1-r_2)+2r_2=0 \text{ or } r_1=-\frac{\overline{2-\mu}}{\mu}\,r_2$$

If
$$\mu=\frac{3}{2}\qquad r_1=-\frac{r_2}{3}$$

We may also find from the general equation the appropriate value of u for a lens having one plane side.

Suppose the front plane, then writing the equation
$$\frac{2r_2-d}{u}=\mu-\frac{A}{r_1} \text{ where A is finite}$$

we see that when r_1 is infinite
$$u=\frac{2r_2-d}{\mu}$$

and if
$$\mu=\frac{3}{2} \text{ and } d \text{ negligible}$$
$$u=\frac{4r_2}{3}$$

as the focal length f is in this case equal to $-2r_2$, nearly,
$$u=-\frac{2}{3}f, \text{ about.}$$

If the lens is convex, and f negative, this implies that the object should be in front of the lens, at $\frac{2}{3}$ of the numerical value of the focal length from it.

It should be at the point $\frac{1}{3}$ of the diagram.

If the lens is concave it should be employed upon a convergent pencil, converging to a point $\frac{2}{3}f$ behind the lens. This point is again the point $\frac{1}{3}$ of the diagram.

NOTE.—It should be carefully remembered that focal length diminishing generally with increase of curvature, the thickness of a convex lens rises in importance as focal length diminishes, and must not be neglected in cases of short focal length, such as the objectives of microscopes.

Suppose a hemisphere, with its base turned towards an object. Here $d = r_2$ and r_1 is infinite and $u = \frac{r_2}{\mu}$.

To find the condition that a plano-convex lens shall have, the principal focus on the plane side coincident with the point in which the object must be placed in order that the rays may pass with minimum deviation through the lens.

In this case
$$f = -\frac{r_2}{\mu - 1}$$
$$r_1 = \text{infinity}$$
$$u_0 = \frac{d}{\mu} - f$$

and the value of u the distance of the object to fulfil the minimum deviation condition is $\frac{2r_2}{\mu} - \frac{d}{\mu}$

If this is equal to u_0

then
$$f = -\frac{2r_2}{\mu}$$
but
$$f = -\frac{r_2}{\mu - 1}$$
∴ also
$$\frac{2}{\mu} = \frac{1}{\mu - 1}.$$
or
$$\mu = 2$$

In conclusion it may be said that all rays pass with minimum deviation through a sphere, since every chord of a sphere cuts the surface at equal angles.

In taking the case of no deviation we must give the value $-\frac{r_1}{r_1 - r_2}d$ to v in the formula for the refraction at one spherical surface, thus:—

Hence
$$\frac{1}{u} = \frac{\mu \overline{r_2 - r_1}}{r_1 d} - \frac{\overline{\mu - 1}}{r_1}$$

$$= \frac{1}{r_1 d}\left\{\mu\,(r_2 - r_1) - \overline{\mu - 1}\;d\right\}$$

$$= \frac{s}{\overline{\mu - 1}\,r_1\,d}$$

$$\therefore u = \frac{r_1 \overline{\mu - 1}\,d}{s}$$

but
$$u_0 = \frac{r_1 \overline{\mu - 1}\,d}{s} - f$$

$$\therefore u = u_0 + f$$

Hence this point is always the point 1 on the diagram, and we have the proof of the proposition that the two points 1 of the diagram of any lens are the images of the centre of the lens for refraction at either of the two surfaces.

CHAPTER XII

ACHROMATIC COMBINATIONS AND DOUBLETS

WE have hitherto considered the facts concerning lenses and their combinations upon the supposition that we have light of only one sort to deal with. The indices of refraction employed are supposed to apply to that particular light. If another kind of light is considered, another index must be used, and we shall obtain different values for (1) the focal length, (2) and (3) the distances of the principal foci from the surfaces of the system. A new diagram in fact exists, and for any position of the object we may have the two images corresponding to the two sorts of light, differing (1) in position upon the axis of the system, which may be called longitudinal chromatic aberration, (2) in magnification, which may also be called transverse chromatic aberration.

The existence of either of these inexactitudes would cause a degree of confusion with light composed of the two sorts. But it might be thought possible so to combine lenses as to do away with either or both of these causes of confusion for two given kinds of light, in which case light of intermediate character would produce very little.

To obtain perfect achromatism for two lights for *all* positions of the object it would be necessary that the two diagrams should coincide in every particular.

A *Doublet* may be defined as a combination of two lenses upon the same axis.

Let there be two lenses whose various properties are symbolized by the letters we have hitherto employed, accents being used in the case of the second lens.

Let k be the distance from the second principal focus of the first lens and the first principal focus of the second lens, positively measured in the direction of light propagation, for one sort of light.

Let D be the distance between the inner or opposed faces of the lenses.

And let U and V be the distances from the first and last (of four) surfaces of the system of the final first and second principal foci of the system for one of the sorts of light, and F the final focal length.

Further, let x be the distance of an object from the first surface of the first lens, positively measured against the direction of the light, and let m be the magnification of its image,

And y the distance of the image from the last surface of the last lens, positively measured against the direction of light propagation.

Then we must have

$$k = D - u'_0 + v_0$$

$$m = \frac{F}{x - U}$$

$$(x - U)(y - V) = -F^2 \quad \therefore \quad y = V - \frac{F^2}{(x - U)}$$

$$U = u_0 + \frac{f^2}{k}$$

$$V = v'_0 - \frac{f'^2}{k}$$

$$F = \frac{ff'}{k} = \frac{ff'}{D - u'_0 + v_0} \quad \text{or} \quad \frac{1}{F} = \frac{D - u'_0 + v_0}{ff'}$$

Since we can express y m and F in terms of U V F D, and U V F can be expressed in terms of the optical elements of the two lenses, themselves depending upon the structural elements of those lenses including the indices of refraction, and the distance D, therefore y m and F can be expressed in terms of quantities the only variables among which are the two indices of refraction.

Hence the condition that the focal length shall be the same for the two lights is:—

$$\frac{d}{d\mu}\frac{1}{F}\,\delta\mu+\frac{d}{d\mu'}\frac{1}{F}\,\delta\mu'=0$$

where $\mu+\delta\mu$ is the index of refraction for the second sort of light for the material of the first lens, $\mu'+\delta\mu'$ the same for the material of the second lens.

Similarly for the magnification to remain the same,

$$\frac{d\frac{1}{m}}{d\mu}\,\delta\mu+\frac{d\frac{1}{m}}{d\mu'}\,\delta\mu'=0$$

and for no longitudinal aberration

$$\frac{dy}{d\mu}\,\delta\mu+\frac{dy}{d\mu'}\,\delta\mu'=0$$

The process of carrying out the exact investigation would be excessively laborious, and it is generally considered sufficient to shorten the labour by considering the lenses as indefinitely thin.

Adopting this course we shall have in the case of the first lens (making $d=0$),

$$s=\mu.\overline{\mu-1}\ \overline{r_2-r_1}$$

$$\frac{1}{f}=\overline{\mu-1}\ \frac{\overline{r_2-r_1}}{r_2r_1},\ \text{whence}\ \frac{d\frac{1}{f}}{d\mu}=\frac{1}{\overline{\mu-1}\ f}$$

$$u_0=-f$$

$$v_0=f$$

Similarly for the second lens,

$$\frac{1}{f'}=\overline{\mu'-1}\ \frac{\overline{r'_2-r'_1}}{r'_2r_1'},\ \text{whence}\ \frac{d\frac{1}{f'}}{d\mu'}=\frac{1}{\overline{\mu'-1}}\,\frac{}{f'}$$

$$u'_0 = -f'$$

$$v'_0 = f'$$

and $$k = \mathrm{D} + f + f'$$

$$\mathrm{U} = \frac{f^2}{k} - f = \frac{-f\,\overline{\mathrm{D}+f'}}{\mathrm{D}+f+f'}$$

$$\mathrm{V} = -\frac{f'^2}{k} + f' = \frac{f'\,\overline{\mathrm{D}+f}}{\mathrm{D}+f+f'}$$

$$\frac{1}{\mathrm{F}} = \frac{\mathrm{D}+f+f'}{ff'} = \frac{\mathrm{D}}{ff'} + \frac{1}{f} + \frac{1}{f'}$$

$$\frac{d\frac{1}{\mathrm{F}}}{d\mu} = \frac{\mathrm{D}}{f'f}\,\frac{1}{\overline{\mu-1}} + \frac{1}{f\,\overline{\mu-1}} = \frac{\mathrm{D}+f'}{ff'\,\overline{\mu-1}}$$

$$\therefore \frac{d\frac{1}{\mathrm{F}}}{d\mu}\,\delta\mu = \frac{\overline{\mathrm{D}+f'}}{ff'\,\overline{\mu-1}}\,\delta\mu$$

and $$\frac{d\frac{1}{\mathrm{F}}}{d\mu'}\,\delta\mu' = \frac{\mathrm{D}+f}{ff'\,\overline{\mu'-1}}\,\delta\mu'$$

$$\therefore \frac{d\frac{1}{\mathrm{F}}}{d\mu}\,\delta\mu + \frac{d\frac{1}{\mathrm{F}}}{d\mu'}\,\delta\mu' =$$

$$\frac{\overline{\mathrm{D}+f}\;\overline{\mathrm{D}+f'}}{f\,f'}\left\{\frac{\delta\mu}{\overline{\mathrm{D}+f}\;\overline{\mu-1}} + \frac{\delta\mu'}{\overline{\mathrm{D}+f'}\;\overline{\mu'-1}}\right\}$$

Let us call this function H as an abbreviation.

The condition that the focal length shall be unchanged ($\mathrm{H} = 0$) is therefore

$$\frac{\frac{\delta\mu}{\mu-1}}{\frac{\delta\mu'}{\mu'-1}} = -\frac{\mathrm{D}+f}{\mathrm{D}+f'}$$

In order that the magnification of the image of the object whose position is given by x, may remain the same for both lights we must operate upon the function expressing $\frac{1}{m}$ or $\frac{x-U}{F}$

Now $$\frac{U}{F}=-\frac{D+f'}{f'}=-\frac{D}{f'}-1$$

$$\therefore \frac{d\frac{1}{m}}{d\mu}\,\delta\mu+\frac{d\frac{1}{m}}{d\mu'}\,\delta\mu'=xH+\frac{D}{f'\overline{\mu'-1}}\,\delta\mu'\equiv M$$

and M must be equal to zero for transverse achromatism. For the longitudinal displacement we must operate in the above way upon y or $V-\frac{F^2}{x-U}$

This function is equal to

$$F\left\{\frac{V}{F}-\frac{F}{x-U}\right\}$$

differentiating

$$\left(\frac{V}{F}-\frac{F}{x-U}\right)dF+F\left\{d\,\frac{V}{F}-d\,\frac{F}{x-U}\right\}$$

$$=-\left(\frac{V}{F}-\frac{F}{x-U}\right)F^2\,d\,\frac{1}{F}+F\left\{d\,\frac{D+f}{f}+\frac{F^2}{(x-U)^2}\,d\,\frac{x-U}{F}\right\}$$

$$=-\left(\frac{V}{F}-\frac{F}{x-U}\right)F^2H+F\left\{\frac{D}{\overline{\mu-1}}\,\frac{\delta\mu}{f}+\frac{F^2}{(x-U)^2}\left(xH+\frac{D}{f'\overline{\mu'-1}}\,\delta\mu'\right)\right\}$$

$$=\left[-\left(V-\frac{F^2}{x-U}\right)H+\frac{F^2x}{(x-U)^2}\,H+D\left\{\frac{d\mu}{f\overline{\mu-1}}+\frac{F^2}{(x-U)^2}\,\frac{\delta\mu'}{f'\,\overline{\mu'-1}}\right\}\right]F$$

But $$V-\frac{F^2}{x-U}=\frac{Vx-(UV+F^2)}{x-U}$$

and $$UV=-\frac{ff'\,\overline{D+f}\,\overline{D+f'}}{(D+f+f')^2}$$

$$=-\frac{F^2\,\overline{D+f}\,\overline{D+f'}}{f\,f'}$$

$$\therefore\ \mathrm{UV}+\mathrm{F}^2=\mathrm{F}^2\left\{1-\frac{\overline{\mathrm{D}+f}\ \overline{\mathrm{D}+f'}}{f\,f'}\right\}$$

$$=-\frac{\mathrm{F}^2\mathrm{D}}{ff'}\left\{\mathrm{D}+f+f'\right\}$$

$$=-\mathrm{FD}$$

$$\therefore\ \mathrm{V}-\frac{\mathrm{F}^2}{x-\mathrm{U}}=\frac{\mathrm{V}x+\mathrm{FD}}{x-\mathrm{U}}$$

Therefore the longitudinal aberration is

$$\left\{\frac{\mathrm{F}^2}{\overline{x-\mathrm{U}}^2}\left(x\mathrm{H}+\frac{\mathrm{D}}{f'}\,\frac{\delta\mu'}{\overline{\mu'-1}}\right)-\frac{\mathrm{V}x+\mathrm{FD}}{\overline{x-\mathrm{U}}}\ \mathrm{H}+\mathrm{D}\ \frac{\delta\mu}{f\,\overline{\mu-1}}\right\}\mathrm{F}\equiv\mathrm{LF}$$

where
$$\mathrm{L}=\frac{\mathrm{F}^2}{(x-\mathrm{U})^2}\ \mathrm{M}-\frac{\mathrm{V}x+\mathrm{FD}}{x-\mathrm{U}}\ \mathrm{H}+\mathrm{D}\ \frac{\delta\mu}{f\,\overline{\mu-1}}$$

This equation and the two

$$\mathrm{M}=x\mathrm{H}+\frac{\mathrm{D}}{f'\,\overline{\mu'-1}}\ \delta\mu'$$

$$\mathrm{H}=\frac{\overline{\mathrm{D}+f}\ \overline{\mathrm{D}+f'}}{ff'}\left\{\frac{\delta\pi}{\overline{\mathrm{D}+f}\ \overline{\mu-1}}+\frac{\delta\mu'}{\overline{\mathrm{D}+f'}\ \overline{\mu'-1}}\right\}$$

enable us to discuss the general question of achromatism for any doublet.

It is not unusual to denote $\frac{\delta\mu}{\mu-1}$ by a simple symbol ω to which the name dispersive power at the particular value μ of the index of refraction is given.

It is never anything but a small proper fraction, and will always have to be compared with similar quantities in settling questions of achromatism. Its value depends upon the particular rays we have to deal with, and is formed by dividing the difference of the indices for the two rays by the index for either of them, or the mean of the two indices, diminished by unity.

We can therefore write the foregoing equations thus :—

$$\mathrm{H}=\frac{\overline{\mathrm{D}+f}\,\overline{\mathrm{D}+f'}}{ff'}\left\{\frac{\omega}{\overline{\mathrm{D}+f}}+\frac{\omega'}{\overline{\mathrm{D}+f'}}\right\}$$

$$\text{M} = x\text{H} + \frac{\text{D}\omega'}{f'}$$

$$\text{L} = \frac{\text{F}^2}{(x-\text{U})^2}\text{M} - \frac{\text{V}x+\text{FD}}{x-\text{U}}\text{H} + \frac{\text{D}\omega}{f}$$

If the material of the lenses is identical, $\omega = \omega'$, and the equations become

$$\text{H}_1 = \frac{\omega}{ff'}\left\{2\text{D} + f + f'\right\}$$

$$\text{M}_1 = \frac{\omega}{ff'}\left\{x\,(2\text{D} + f + f') + f\text{D}\right\}$$

$$\text{L}_1 = \frac{\text{F}^2}{(x-\text{U})^2}\text{M}_1 - \frac{\text{V}x+\text{FD}}{x-\text{U}}\text{H}_1 + \frac{\text{D}\omega}{f}$$

Suppose that $\text{H} = 0$, then M cannot be equal to zero unless also $\text{D} = 0$, in which case the condition of H being zero becomes simplified into

$$\frac{\omega}{f} + \frac{\omega'}{f'} = 0 \quad . \quad . \quad . \quad . \quad (a)$$

Then when M H and D are zero L also is zero and the combination achromatic completely. The conditions for this, therefore, are contained in the equation (*a*) and in $\text{D} = o$.

The thickness of the lenses having been considered to vanish, the points 1 of the diagram of each lens coincide with the lenses themselves. The equation $\text{D} = o$ therefore implies that the point 1 in the diagram of the first lens for light which has passed through it, shall coincide with the point 1 in the diagram of the second lens for light which is approaching it. A common construction of an achromatic combination of two lenses consists in a convex lens combined with a plano-concave. The latter has a longer focal length than the former, but is composed of glass which has a greater dispersive power ω' than that of the convex lens, ω, the intention being to obey the equation (*a*) written in the form :—

$$\frac{f'}{f} = -\frac{\omega'}{\omega}$$

Now a lens having one side plane, whether convex or concave, will always have the point 1 for light approaching its curved side, on that curved surface.

The second lens (concave) is easily arranged to be of this character, and the convex lens may touch it. But the convex lens, when made of the less refracting material, to have a shorter focal length is generally made a double convex. This prevents the point 1 for light between the lenses being on the surface, but causes it to be sunk a certain distance within the material of the lens.

It is not, however, necessary that the two surfaces in contact should have the same curvature, though this is convenient when the two lenses are cemented together, both for mechanical firmness and for avoiding any refraction from the cementing material, which would in that case be extremely thin and uniform in thickness measured parallel to the axis.

We shall therefore consider the case of a plano-convex lens and a concave-plane lens placed in contact at the curved sides.

Let r be the radius of the curved side of the convex
and ρ ,, ,, ,, ,, concave.

Consulting the fundamental equations and making $r_1 = \rho_2 =$ infinity, we find that

$$f = -\frac{r}{\mu - 1}$$

$$f' = \frac{\rho}{\mu' - 1}$$

and the equation of condition becomes

$$\frac{\dfrac{\rho}{\mu' - 1}}{\dfrac{r}{\mu - 1}} = \frac{\dfrac{\delta\mu'}{\mu' - 1}}{\dfrac{\delta\mu}{\mu - 1}}$$

or

$$\frac{\rho}{r} = \frac{\delta\mu'}{\delta\mu}$$

The distance k from the second principal focus of the first lens to the first principal focus of the second lens is $f' + f$, or $\left\{\frac{\rho}{\mu' - 1} - \frac{r}{\mu - 1}\right\}$ a positive quantity. Therefore the focal length of the combination is

$$\frac{ff'}{\frac{\rho}{\mu' - 1} - \frac{r}{\mu - 1}}$$

$$\text{or} \qquad \frac{f\omega'}{\omega' - \omega}$$

This is the focal length which will result from achromatizing a plano-convex lens of a glass defined by the quantities μ and $\delta\mu$, worked to radius r, by means of a lens of glass defined by the quantities μ' and $\delta\mu'$, and the radius of curvature to be given to the concave face of the second lens is given by $\rho = r\frac{\delta\mu'}{\delta\mu}$

The only real meaning which can be attached to the idea that the focal length does not change for a change in ray is that a maximum or minimum value of the focal length obtains under the circumstances. In any other sense the statement carries absurdity on the face of it.

A similar remark applies to the cases of no change in the position or magnification of the image.

It should be mentioned that in achromatic object glasses of telescopes, of two materials, it is considered better to make the surfaces in contact of the same radius, and cement them together. In this case D is neglected (though it is evident it cannot actually vanish) and the lens of less refracting glass made double convex.

Recurring to the equation for H we may observe that this is equal to zero even when the glasses are of the same character, when

$$2\,\mathrm{D} + f + f' = 0$$

or
$$\text{D} = -\frac{f+f'}{2}$$

Now this is the case in what is called Huyghen's eye-piece, which is constructed of two convex lenses whose focal lengths are as $-3 : -1$ numerically placed at a distance apart bearing to these numbers the relation 2.

It is therefore commonly described as achromatic, but since D is finite and H vanishes, M cannot do so.

As single lenses are not achromatic, it is a matter of importance in practical optics to study how the defects of simple lenses may be remedied by the employment of combinations. There are other drawbacks in simple lenses due to the fact that in practice strictly central pencils of light are not solely employed. When two lenses are employed as a substitute for one, the combination is called a doublet. The eye-pieces of telescopes and microscopes are generally doublets. Thus we have Ramsden's eye-piece, Huyghen's eye-piece, Wollaston's doublet, etc., combinations which are found to give good results.

These doublets or eye-pieces, of two lenses of similar glass, are denfied by the two focal lengths and the distance between the points 1 of magnification for light between them. We may call them f_1 and f_2 in the order in which light encounters them and D. In thin lenses D would be equal to the distance between the lenses, actually it would depend upon the thickness of the individual lenses. If we symbolize the resulting focal length by F, and the distances of the principal foci from the surfaces by U_0 V_0, we may give categorically the details of a few such combinations, in terms of D $(=1)$.

Wollaston's doublet—

$$f_1 = -\frac{2}{3} \qquad f_2 = -2 \qquad \text{F} = -\frac{4}{7} \qquad \text{U}_0 = \frac{10}{21} \qquad \text{V}_0 = -\frac{2}{7}$$

Ramsden's eye-piece—

$$f_1 = -\frac{3}{2} \qquad f_2 = -\frac{3}{2} \qquad \text{F} = -\frac{9}{8} \qquad \text{U}_0 = \frac{3}{8} \qquad \text{V}_0 = -\frac{3}{8}$$

Huyghen's eye-piece—

$$f_1 = -\frac{3}{2} \qquad f_2 = -\frac{1}{2} \qquad F = -\frac{3}{4} \qquad U_0 = -\frac{3}{4} \qquad V_0 = -\frac{1}{4}$$

A combination of two equal convex lenses placed so that one is in the principal focus of the other—

$$f_1 = -1 \qquad f_2 = -1 \qquad F = -1 \qquad U_0 = 0 \qquad V_0 = 0$$

A combination giving a very well-defined view at the centre of the field, but at the expense of some aberration towards the edge, is—

$$f_1 = -\frac{4}{3} \qquad f_2 = -\frac{1}{3} \qquad F = -\frac{2}{3} \qquad U_0 = -\frac{4}{3} \qquad V_0 = -\frac{1}{6}$$

A good eye-piece is found to be the following—

$$f_1 = -\frac{12}{5} \qquad f_2 = -\frac{8}{5} \qquad F = -\frac{32}{25} \qquad U_0 = \frac{12}{25} \qquad V_0 = -\frac{56}{75}$$

In interpreting these values it must be carefully remembered that if U_0 is positive, the principal focus for light before encountering the combination is outside the surface, and *vice versâ.*

If V_0 is negative the principal focus for light emerging from the combination is outside the surface, and *vice versâ.* Of the combinations mentioned above the Huyghen's eye-piece and the next one are achromatic in the sense that

$$-\frac{f_1 + f_2}{2} = D$$

CHAPTER XIII

THE MEASUREMENT OF THE INDEX OF REFRACTION BY GONIOMETER

We have indicated how the index of refraction of a transparent substance may be found if the substance is in the form of a lens of convex character. If the substance is in the form of a prism, an incomparably more accurate determination may be made. The method depends upon measuring the minimum deviation of light passing through a prism, a condition, like all maxima and minima conditions, more easy to arrive at than other cases.

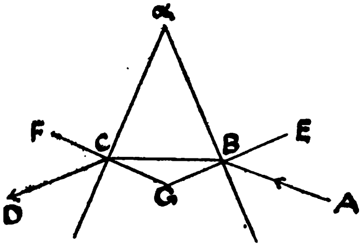

Fig. 30.

The passage with minimum deviation must necessarily be a symmetrical one. Let A B C D be a ray passing through a prism in a principal plane with minimum deviation. And let the refracting angle of the prism be a. Then B C is the base of an isosceles triangle whose equal sides are a C, a B.

At B and C draw straight lines E B G, F C G, perpendicular to the surfaces of the prism at B and C, meeting in G.

Then clearly the angles GBC, GCB are each equal to $\frac{a}{2}$, and the sine of the angle ABE must be equal to $\mu \sin \frac{a}{2}$. But the deviation of the ray at B must be equal to ABE − GBC, and the deviation at C must be the same in amount; therefore calling d the total deviation of the ray,

$$\frac{d}{2} = \text{ABE} - \frac{a}{2}$$

$$\therefore \text{ABE} = \frac{d+a}{2}$$

$$\therefore \sin \text{ABE} = \sin \frac{d+a}{2} = \mu \sin \frac{a}{2}$$

Hence
$$\mu = \frac{\sin \frac{d+a}{2}}{\sin \frac{a}{2}}$$

If therefore we can determine the angles d and a we have by means of this simple formula the value of μ.

Instruments for determining angles are called goniometers. When the rays of different character in the spectrum may be individually studied it is called a spectrometer, or spectroscope.

A goniometer consists of a collimator B and a telescope C. The latter is capable of moving round a fixed point A a few inches in front of the object glass. It has spider lines in the common principal focus of its object glass and eye-piece, for it must be focussed for infinity.

The collimator consists of a tube carrying a convex lens and a fine slit in its principal focus. The length of the slit must be parallel to the axis through A, about which the telescope turns. The adjustment of the slit in the principal focus of the object glass may be tested by turning the telescope, focussed for infinity, so as to be in a line with the collimator. Then to an eye looking through the telescope the slit must appear perfectly sharp and well defined. If this is not at first the case the length of the collimator between lens and slit must be adjusted until good definition is secured.

Sometimes in experimenting with different kinds of light, both the telescope and collimator have to be readjusted to meet such changes. We have shown how to put a telescope into the telescopic condition for any sort of light.

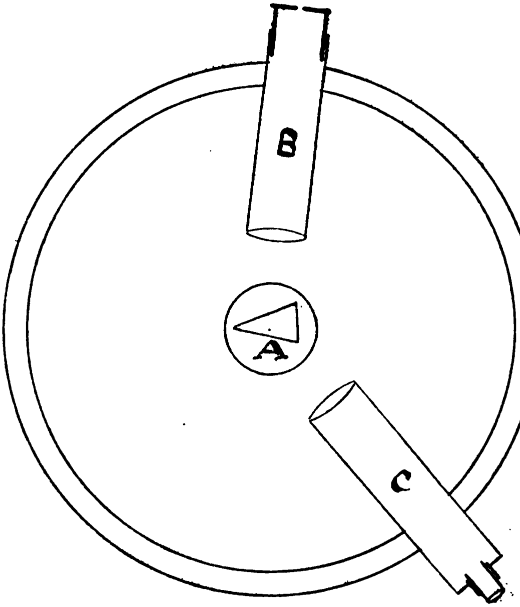

FIG. 31.

At the point A is a small table turning independently round the axis of the motion of the telescope. The top of this table should be truly plane and have the axis at A normal to it.

The position of the telescope at any moment is determined from some zero point by a vernier which it carries over the graduated circle. The collimator must remain fixed relatively to this graduated circle.

To observe the minimum deviation, place the prism at A with its refracting edge vertical, and search for the image of the slit by moving the telescope. Move the table A this way and that way, following the image with the telescope so long as it is necessary to move the telescope more nearly into line with the collimator, for it is clear that when minimum deviation exists, the ray and the telescope following the ray cannot be brought closer to the line of the collimator than the angle of minimum deviation.

When the adjustment is perfect a motion of the table at A in either direction will bring the image of the slit into coincidence with the spider lines, but will not cause it to cross them.

This being settled and the reading in angle taken, the reading of the vernier when the telescope is pointed exactly into the collimator may be taken. The difference of the two readings is the minimum deviation.

Or the following course may be adopted. Point the prism towards the other direction, and again find the position of minimum deviation, and read the vernier.

The difference between the two readings is now twice the deviation, which is therefore determined.

To determine the refracting angle a of the prism, place it so that its refracting angle points about straight to the collimator.

Then clearly it will receive rays on either face, and the telescope may be moved to right and left and receive in either case a reflected image at its spider lines.

The difference of reading in the two cases will be obviously twice the angle of the prism, or 2 a ., and with the minimum deviation serves for the evolution of μ in the formula

$$\mu = \frac{\sin \frac{d+a}{2}}{\sin \frac{a}{2}}$$

A caution may here be introduced for the sake of those who attempt such measurements for the first time. Though we define a prism to be a mass of refracting material contained between *two* plane surfaces, meeting at an angle, it often happens in practice that prisms are masses of glass having *three* plane surfaces, intersecting in parallel straight lines. It is well in making measurements to cover the side which is not to be employed with paper, or with crystallized syrup. Otherwise images may occur produced by the third side, which may be mistaken for others. For instance, in the figure, the images produced by the two courses of rays A B C D, A F G, may be mistaken one for the other.

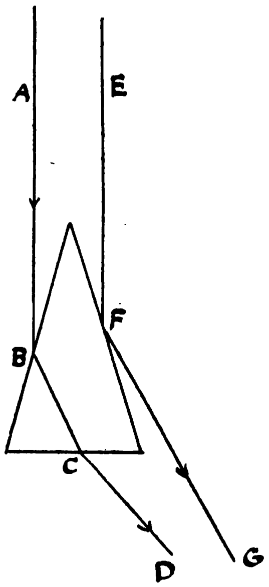

FIG. 32.

The following suggestions will, however, serve to prevent mistakes of this sort :—

The slit is very generally made movable on one side, so that a very fine slit may be employed when it is wanted.

Turn the telescope to the collimator and move the screw actuating the side of the slit. It will then be noticed which side of the slit appears movable and which fixed. Suppose it be the right hand that appears fixed.

Then in the refracted ray, when the minimum deviation is being determined, the right side of the slit will also appear fixed, but in an image produced by reflection, the left-hand side of the slit would appear fixed. This will generally prevent a misunderstanding, but as the slit is varied in dimensions during the observations, it would not do to make the spider lines bisect the slit. In this case, therefore, the spider lines must be brought to coincidence with the stationary side of the slit. Another advantage from this course is that the slit may be left pretty wide, and so the general illumination increased. When the slit is very fine, with feeble light, it is often not possible to detect the spider lines.

*

APPENDIX

A method for finding the angular value of the diameter of the field of view of a telescope.

LET two circles of radii R, r, move relatively to one another in such a way that one crosses the other in a straight line, and at some part of the passage is entirely contained by the other.

There will then be necessarily four contacts, two external and two internal, between the two circles. In the figure

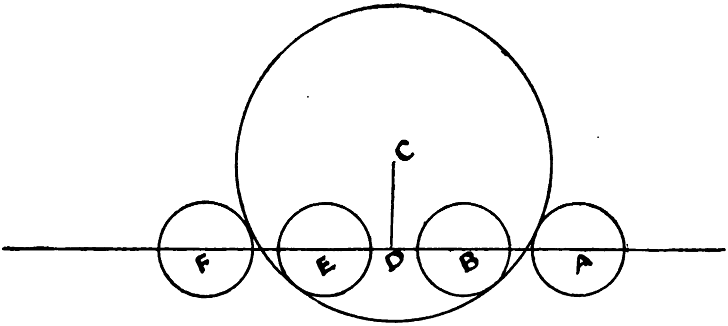

FIG. 33.

A, B, E, F are the positions of the centre of one circle at the four contacts. C is the centre of the other circle. C D is a perpendicular from C upon the path of the centre of the moving circle.

Let v be the velocity of the motion.

,, t_1 ,, interval of time between external contacts.

,, t_2 ,, ,, ,, ,, internal ,,

$$\text{Then } (R+r)^2 = CA^2 = AD^2 + CD^2,$$
$$\text{and } (R-r)^2 = CB^2 = BD^2 + CD^2.$$

Hence by subtraction—

$$4\,Rr = AD^2 - BD^2.$$

But $$AD = \frac{v\,t_1}{2} \text{ and } BD = \frac{v\,t_2}{2}.$$

Hence $$Rr = \frac{v^2}{16}\left(t_1^2 - t_2^2\right) \quad . \quad . \quad (a).$$

Now suppose that R is the angular value of the radius of the field of view of the telescope, and that D $(=2\,R)$ is the angular value of the diameter. Also that r is the angular value of the semi-diameter of the sun, and that T is the interval of time which the semi-diameter of the sun takes in crossing a meridian due to the rotation of the earth. Then $v\,T = r$ and the equation (a) becomes—

$$D = \frac{r}{8}\left(\frac{t_1^2 - t_2^2}{T^2}\right)$$
$$= \frac{r}{2}\,\frac{t_1+t_2}{2}\,\frac{t_1-t_2}{2}\,\frac{1}{T^2} \quad . \quad . \quad (\beta).$$

Now $\frac{t_1+t_2}{2}$ is the interval of time (t_3) between the first and third contact, or between the second and fourth, and $\frac{t_1-t_2}{2}$ is the interval of time (t_4) between the first and second contact, or between the third and fourth.

Hence $$D = \frac{r}{2}\,\frac{t_3 t_4}{T^2} \quad . \quad . \quad . \quad . \quad . \quad (\gamma).$$

Of the quantities involved in this formula

r and T are given in the *Nautical Almanack*,
t_3 and t_4 must be deduced from observation.

In any actual case the first contact would be less accurately observed than any of the other three contacts, but it will be noted that it is unnecessary to observe the time at more than three of the contacts.

The *Nautical Almanack* gives the interval T in sidereal time, and t_3t_4 should of course be measured in the same units. If, however, t_3t_4 are in solar time it will be necessary to increase the value of D obtained by 5·5 per thousand.

It is scarcely necessary to add that any circle in the field of view, such as a ring micrometer, may have its diameter determined in this way.

The following example, taken from actual experiment, may illustrate the working of this method.

Sept. 18, 1902.

	h.	m.	sec.	
Time of first inner contact	5	52	54	p.m. by chronometer.
,, second ,, ,,	5	53	52·5	,, ,,
,, last contact	5	56	2·5	,, ,,

Hence t_3 = 3 m. 8·5 sec. = 188·5 sec. solar interval.
t_4 = 2 m. 10 sec. = 130· sec. ,, ,,
from *Nautical Almanack* T = 64 sec. sidereal ,,
,, ,, ,, sun's semi-diameter = 15′ 56·″91
= 956·″91.

Hence D (in minutes of arc)

$$= \frac{478{\cdot}455 \times 188{\cdot}5 \times 130 \times 1{\cdot}0055}{60 \times 64^2}$$

$$= 47'{\cdot}97.$$

ULAR CASES.

$-r_1=r_2=r=\frac{d}{2}$ sphere.	r_1 infinite. $d=r_2$ hemisphere ◖.	r_2 infinite. $d=-r_1$ hemisphere ◗.	$s=2r_2\,\overline{\mu-1}$
$-\frac{2\,\overline{\mu-1}}{r}$	$-\frac{\mu\,\overline{\mu-1}}{r_2}$	$\frac{\mu\,\overline{\mu-1}}{r_1}$	$\frac{2\,\overline{\mu-1}}{r_1}$
$-\frac{\mu r}{2\,\overline{\mu-1}}$	$-\frac{r_2}{\mu-1}$	$\frac{r_1}{\mu-1}$	$\frac{\mu r_1}{2\,\overline{\mu-1}}$
$-\frac{r}{2}\,\frac{\mu-2}{\mu-1}$	$\frac{r_2}{\mu\,\overline{\mu-1}}$	$-\frac{r_1}{\mu-1}$	$\frac{r_1 d}{2r_2}-\frac{\mu r_1}{2\,\overline{\mu-1}}$
$\frac{r}{2}\,\frac{\mu-2}{\mu-1}$	$-\frac{r_2}{\mu-1}$	$\frac{r_1}{\mu\,\overline{\mu-1}}$	$\frac{d}{2}+\frac{\mu r_1}{2\,\overline{\mu-1}}$
any value	$\frac{r_2}{\mu}$	$-\frac{2r_1}{\mu-2}$	infinite
any value	$-\frac{\mu-2}{\mu}$	$-\frac{\mu}{\mu-2}$	infinite
$-r$	$-r_2$	0	
any value	r_2	$2r_1$	

mentioned in lines 5, 7, and 8, when positive are to be measured
ith minimum deviation.

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